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Extremal graphs with bounded vertex bipartiteness number

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A B S T R A C T

Given a graph G. The fewest number of vertices whose deletion yields a bipartite graph from G was defined by S. Fallat and Yi-Zheng Fan to be the *vertex bipartiteness* of G and it is denoted by $\upsilon_b(G)$. We consider the set $\Sigma_k(n)$ defined by

$$\{G = (V(G), E(G)) : G \text{ connected}, \quad |V(G)| = n \text{ and } \upsilon_b(G) \le k\}.$$

In this work we identify the graph in $\Sigma_k(n)$ with maximum spectral radius and maximum signless Laplacian spectral radius.

1. Introduction

Let $G = (V(G), E(G))$ be an undirected simple graph with n vertices and m edges. Usually, we consider that the graph G has order n, that is $V(G) = \{1, \ldots, n\}$. The set

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$E(G)$ is the set of edges of G. An edge with end vertices i and j is denoted by ij and we say that the vertices i and j are adjacent or neighbors. The multiset of the eigenvalues of an arbitrary square matrix, M is the spectrum of M and will be denoted by σ_M. More generally, $\sigma_M = \{\tau_1^{[m_1]}, \tau_2^{[m_2]}, \ldots, \tau_q^{[m_q]}\}$ denotes that τ_1 has multiplicity m_1, τ_2 has multiplicity m_2, and so on. If M is a symmetric matrix of order n its eigenvalues are ordered as follows: $\tau_1 \geq \cdots \geq \tau_n$. The spectral radius of M is

$$\rho_M = \max\{|\tau| : \tau \in \sigma_M\}.$$

The eigenvalues of G, $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ are the eigenvalues of its adjacency matrix, A_G. The spectrum of G, is σ_{A_G}, which we abbreviate as σ_G. If G has at least one edge, then A_G has a negative eigenvalue, not greater than -1 and a positive eigenvalue not less than the average degree of the vertices of G (see [2,10]). The number of neighbors of a vertex i is the degree of i and the neighborhood of i, $N_G(i)$ is the set of its neighbors. The maximum and minimum degree of the vertices of G are $\Delta(G)$ (or Δ) and $\delta(G)$ (or δ). A graph G is called p-regular whenever $\Delta = \delta = p$. A subgraph H of G is an induced subgraph if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. Thus, an induced subgraph is determined by its vertex set. Indeed, by deleting some vertices of G together with the edges incident to those vertices we obtain an induced subgraph. The induced subgraph with vertex set $S \subset V(G)$ is denoted by $\langle S \rangle$. A bipartite graph is a graph G whose vertex set can be divided into two disjoint set X, Y such that every edge has an end vertex in X and the other one in Y. The set $\{X, Y\}$ is called a bipartition of G. It is well known that if G is a bipartite graph then $\lambda_n(G) = -\lambda_1(G)$. A complete bipartite graph $K_{s,t}$ is a bipartite graph with a bipartition $\{X, Y\}$, where $|X| = s$ and $|Y| = t$ and such that any two vertices $i \in X$ and $j \in Y$ are connected by an edge. The complete graph K_n is a graph on n vertices such that any two distinct vertices are connected by an edge. Let G_1 and G_2 be two vertex-disjoint graphs. The *joinof* G_1 and G_2 is the graph $G_1 \vee G_2$ such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{ij : i \in V(G_1) \text{ and } j \in V(G_2)\}$.

Let D_G be the diagonal matrix of vertex degrees. It is a consequence of the Geršgorin Theorem (see [14]) that the matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are positive semidefinite. We refer to these matrices as the Laplacian and the signless Laplacian matrix of G, respectively and their spectrum are the Laplacian and signless Laplacian spectrum of G, respectively. The spectrum of Q_G equals that of L_G if and only if G is a bipartite graph (see [3,4,8,9]). The minimum number of vertices (resp., edges) whose deletion yields a bipartite graph from G is called the *vertex bipartiteness* (resp., *edge bipartiteness*) of G and it is denoted $\upsilon_b(G)$ (resp., $\epsilon_b(G)$), see [6]. Let s_n be the smallest eigenvalue of Q_G, S. Fallat and Yi-Zheng Fan in [6] established the inequality:

$$s_n \leq \upsilon_b(G) \leq \epsilon_b(G). \tag{1}$$

There are numerous results about the largest eigenvalue of L_G, the spectral radius, namely see [1,5,11,12] for upper and lower bounds. In the present paper we obtain an