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The distance signatures of the incidence graphs of affine resolvable designs

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ABSTRACT

In this note, we determined the distance signatures of the incidence matrices of affine resolvable designs. This proves a conjecture by Kohei Yamada.

1. Introduction

Let A be a $d \times d$ symmetric matrix over the real field $\mathbb{R}$. Let $n_+(A)$, $n_-(A)$, and $n_0(A)$ be the number (counting multiplicity for repeated values) of positive, negative and zero eigenvalues of A, respectively. So $d = n_+(A) + n_-(A) + n_0(A)$. The *signature (inertia)* of A is the triple $(n_+(A), n_-(A), n_0(A))$, denoted by $\operatorname{sig}(A)$.

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For any graph G, we use the same the letter G to denote its vertex set. The distance $\partial(x, y)$ of vertices $x, y \in G$ is the length of a shortest path between them. The distance matrix $D = D(G)$ is formed by indexing the rows and columns with the vertex set G and defining the (x, y) entry to be $\partial(x, y)$. Following [5], the signature $\mathrm{sig}(D)$ is called the *distance signature* of graph G. We also write $\mathrm{sig}(G)$ for $\mathrm{sig}(D)$ and $n_{\pm,0}(D)$ for $n_{\pm,0}(G)$. Since D is real and symmetric, it has real eigenvalues. Because the trace of $D(G)$ is zero, $n_+(G)$ and $n_-(G)$ are bounded above by $|G| - 1$.

In [5], Graham and Lovász remarked that if it was not even known which of graphs G has $n_-(G) = |G| - 1$ or whether there is a graph for which $n_+(G) > n_-(G)$. Azarija [2] showed that the family of conference graphs has $n_+(G) > n_-(G)$. If G is a conference graph on v vertices, $v \equiv 1 \pmod 4$, then $D(G)$ has a signature $(\frac{v+1}{2}, \frac{v-1}{2}, 0)$. So for $v \geq 9$, $n_+(G) > n_-(G)$.

In this note, we determine the distance signature for the incidence graph of an affine design.

Theorem 1. *Let $\mathscr{D}$ be an affine (v, k, λ)-design with v points and b blocks, where $v = n^2\mu$, $b = (n^3\mu - 1)/(n - 1)$ for integers $n \geq 2$, $\mu \geq 1$. If G is the incidence graph of $\mathscr{D}$, then the distance signature of G is*

$$\mathrm{sig}(G) = \begin{cases} (1, 4, 5) & \text{if } (n, \mu) = (2, 1) \\ (4\mu, 4\mu - 1, 4\mu - 2) & \text{if } n = 2, \mu \geq 2 \\ (b, v, 0) & \text{otherwise.} \end{cases}$$

Recently, Zhang [6] determined the distance signatures of complete k-partite graphs, and Zhang and Godsil [7] gave some graphs with $n_+(G) = 1$ and obtained their distance signatures. See the recent survey paper [1] for more background and activities about the spectra of distance matrices.

2. Preliminaries

A design $\mathscr{D}$ is a pair $(V, \mathcal{B})$, where V is a finite set and $\mathcal{B}$ is a set of subsets of V. Elements of V and $\mathcal{B}$ are called points and blocks, respectively. $\mathscr{D}$ is called a (v, k, λ) design if (1) $|V| = v$, (2) every block contains k points, and (3) every 2-subset of V is contained in precisely λ blocks. For a (v, k, λ) design, it can be shown that the number of blocks containing any fixed point is $\lambda(v - 1)/(k - 1)$, denoted by r. The number of blocks b is vr/k. People sometimes refer $\mathscr{D}$ to as (v, b, r, k, λ) design.

A *parallel class* in $\mathscr{D}$ is a set of blocks that form a partition of V. $\mathscr{D}$ is called a *resolvable* design if $\mathcal{B}$ admits a partition of parallel classes:

$$\mathcal{B} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r.$$

Blocks from the same parallel class are said to be parallel. A resolvable (v, k, λ) design $\mathscr{D}$ is called *affine* if any two distinct non-parallel blocks intersect at exactly μ points.