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An efficient tree decomposition method for permanents and mixed discriminants

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We present an efficient algorithm to compute permanents, mixed discriminants and hyperdeterminants of structured matrices and multidimensional arrays (tensors). We describe the sparsity structure of an array in terms of a graph, and we assume that its treewidth, denoted as ω, is small. Our algorithm requires $\widetilde{O}(n\,2^{\omega})$ arithmetic operations to compute permanents, and $\widetilde{O}(n^2 + n\,3^{\omega})$ for mixed discriminants and hyperdeterminants. We finally show that mixed volume computation continues to be hard under bounded treewidth assumptions.

1. Introduction

The *permanent* of a $n \times n$ matrix M is defined as

$$\mathrm{Perm}(M) := \sum_{\pi} \prod_{i=1}^{n} M_{i,\pi(i)}$$

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where the sum is over all permutations π of the numbers $1, \dots, n$. Computing the permanent is #P-hard [1], which means that it is unlikely that it can be done efficiently for arbitrary matrices. As a consequence, research on this problem tends to fall into two categories: algorithms to approximate the permanent, and exact algorithms that assume some structure of the matrix. This paper lies in the second category. We further study related problems in structured higher dimensional arrays, such as mixed discriminants, hyperdeterminants and mixed volumes.

The sparsity pattern of a matrix M can be seen as the bipartite adjacency matrix of some bipartite graph G. This bipartite graph fully encodes the structure of the matrix. We assume here that the treewidth ω of G is small (constant or logarithmic in n). Even though it is hard to determine the treewidth of a graph, there are many good heuristics and approximations that justify our assumption [2]. In addition, for a fixed k it is possible to check whether $\omega \leq k$ in linear time [3]. We show an algorithm to compute $\mathrm{Perm}(M)$ in $\widetilde{O}(n\, 2^\omega)$ arithmetic operations. In this paper, the notation $\widetilde{O}$ ignores polynomial factors in ω. We note that the algorithm can be used over any commutative ring.

The permanent of a matrix can be generalized in several ways. In particular, given a list of n matrices of size $n \times n$, its *mixed discriminant* generalizes both the permanent and the determinant [4,5]. Our algorithm for the permanent extends in a natural way to compute the mixed discriminant. The natural structure to represent the sparsity pattern in this case is a tripartite (i.e., 3-colorable) graph. The running time of the resulting algorithm is $\widetilde{O}(n^2 + n\, 3^\omega)$, where ω is the treewidth of such graph. In particular, this algorithm can compute the determinant of a matrix in the same time.

More generally, our methods extend to generalized determinants/permanents on tensors. A special case of interest is the *multidimensional permanent* [6–8]. Another interesting case is the first Cayley *hyperdeterminant*, also known as Pascal determinant, which is the simplest generalization of the determinant to higher dimensions [9–11]. Note that unlike the determinant, the hyperdeterminant is #P-hard, in particular because it contains mixed discriminants as a special case [4,12].

Given a set of n polytopes in $\mathbb{R}^n$, its mixed volume provides a geometric generalization of the permanent and the determinant [13]. We focus on the special case of mixed volume of n zonotopes. Although there is no "natural" graph to represent the structure of a set of zonotopes, we associate to it a bipartite graph that, when the mixed volume restricts to a permanent, corresponds to the matrix graph described above. This allows us to give a simple application for mixed volumes of zonotopes with few nonparallel edges. Nevertheless, we show that mixed volumes remain hard to compute in the general case, even if this bipartite graph has treewidth 1 and the zonotopes have only 3 nonzero coordinates.

The diagram of Fig. 1 summarizes the scope of the paper. It presents the main problems we consider, illustrating the relationships among them. Concretely, an arrow from A to B indicates that B is a special instance of A. It also divides the problems according to their difficulty, with and without bounded treewidth assumptions. In this paper we