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Spectral conditions for edge connectivity and packing spanning trees in multigraphs

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ABSTRACT

A multigraph is a graph with possible multiple edges, but no loops. The multiplicity of a multigraph is the maximum number of edges between any pair of vertices. We prove that, for a multigraph G with multiplicity m and minimum degree $\delta \geq 2k$, if the algebraic connectivity is greater than $\min\{\frac{2k-1}{\lceil(\delta+1)/m\rceil}, \frac{2k-1}{2}\}$, then G has at least k edge-disjoint spanning trees; for a multigraph G with multiplicity m and minimum degree $\delta \geq k$, if the algebraic connectivity is greater than $\min\{\frac{2(k-1)}{\lceil(\delta+1)/m\rceil}, k-1\}$, then the edge connectivity is at least k. These extend some earlier results.

A balloon of a graph G is a maximal 2-edge-connected subgraph that is joined to the rest of G by exactly one cut-edge. We provide spectral conditions for the number of balloons in a multigraph, which also generalizes an earlier result.

1. Introduction

A **multigraph** is a graph with possible multiple edges, but no loops. The **multiplicity** of a multigraph is the maximum number of edges between any pair of vertices. In

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this paper, we consider finite undirected multigraphs. Thus, “a graph” in this paper means “a multigraph”, unless otherwise stated. We follow the notations of Bondy and Murty [1] for undefined terms. If $X_1, X_2, \cdots, X_t$ are disjoint vertex subsets of G, then $e(X_1, X_2, \cdots, X_t)$ denotes the number of edges with two endpoints in distinct X_i's, for $i = 1, 2 \cdots, t$. In the following, we assume k and d are positive integers.

Let G be an undirected graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. The **adjacency matrix** of G is an n by n matrix $A(G)$ with entry a_{ij} being the number of edges between v_i and v_j for $1 \leq i, j \leq n$. We use $\lambda_i(G)$ to denote the ith largest eigenvalue of $A(G)$; when the graph G is understood from the context, we use λ_i for $\lambda_i(G)$. With these notations, we always have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $D(G)$ be the **degree matrix** of G, that is, the n by n diagonal matrix with entry a_{ii} being the degree of vertex v_i in G for $1 \leq i \leq n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the **Laplacian matrix** and the **signless Laplacian matrix** of G, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $L(G)$ and $Q(G)$, respectively. It is not difficult to see that $\mu_n(G) = 0$. The second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is known as the **algebraic connectivity** of G.

For a graph G, the **spanning tree packing number**, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A fundamental theorem characterizing graphs G with $\tau(G) \geq k$ has been obtained by Nash-Williams and Tutte.

Theorem 1.1. *(Nash-Williams [11] and Tutte [15]) Let G be a connected graph. Then $\tau(G) \geq k$ if and only if for any partition $(V_1, \ldots, V_t)$ of $V(G)$, $e(V_1, \ldots, V_t) \geq k(t-1)$.*

Cioabă and Wong [4] investigated the relationship between the second largest adjacency eigenvalue and $\tau(G)$ for a regular simple graph G, and made Conjecture 1.1. Utilizing Theorem 1.1, Cioabă and Wong proved the conjecture for $k \in \{2, 3\}$.

Conjecture 1.1. *(Cioabă and Wong [4]) Let k and d be two integers with $d \geq 2k \geq 4$. If G is a d-regular simple graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geq k$.*

Conjecture 1.1 was then extended to Conjecture 1.2 for any simple graph G (not necessarily regular). See [5,6,8,9] for the conjecture and some partial results.

Conjecture 1.2. *(Gu [5], Gu, Lai, Li and Yao [6], Li and Shi [8], Liu, Hong and Lai [9]) Let k be an integer with $k \geq 2$ and G be a simple graph with minimum degree $\delta \geq 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.*

Recently, both Conjectures 1.1 and 1.2 were settled in [10].

Theorem 1.2. *(Liu, Hong, Gu and Lai [10]) Let G be a simple graph with $\delta \geq 2k$.*
(i) If $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(ii) If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(iii) If $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.