Linear Algebra and its Applications 493 (2016) 164–182

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Classification of division gradings on finite-dimensional simple real algebras

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ARTICLE INFO

Article history:
Received 14 July 2015
Accepted 25 November 2015
Available online 17 December 2015
Submitted by R. Brualdi

MSC:
primary 16W50
secondary 16K20, 16S35

Keywords:
Graded algebra
Division grading
Simple real algebra
Classification

ABSTRACT

We classify, up to isomorphism and up to equivalence, division gradings (by abelian groups) on finite-dimensional simple real algebras. Gradings on finite-dimensional simple algebras are determined by division gradings, so our results give the classification, up to isomorphism, of (not necessarily division) gradings on such algebras.

Linear algebra over the field of two elements plays an interesting role in the proofs.

1. Introduction

Consider a finite-dimensional simple real algebra (here by algebra we always mean associative algebra). The Theorem of Artin–Wedderburn states that it is isomorphic to a matrix algebra over a finite-dimensional division real algebra. On the other hand, Frobenius' Theorem characterizes the latter; it says that any finite-dimensional division

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http://dx.doi.org/10.1016/j.laa.2015.11.025
0024-3795/

real algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Both theorems assure the uniqueness, so they classify these algebras.

A graded algebra is said to be a graded division algebra if it is unital and every nonzero homogeneous element has an inverse. In [5, Corollary 2.12] (see also [2,3,7]) an analogue of Artin–Wedderburn's Theorem for graded simple algebras is proved: they are, roughly speaking, graded matrix algebras over graded division algebras. This gives a classification up to isomorphism of gradings. The difficulty of studying the equivalence classes if the grading is not fine is illustrated in [5, Examples 2.40 and 2.41]. The classification will be completed if an analogue of Frobenius' Theorem for gradings is given, that is, if the graded division algebras are classified. This is done in [5, Theorem 2.15] for the case in which the ground field is algebraically closed; in this paper we will do it for the real case. Therefore, the results in this text give a complete classification of the isomorphism classes of gradings on finite-dimensional simple real algebras.

The goal of this text is to classify, up to isomorphism and up to equivalence, division gradings (by abelian groups) on finite-dimensional simple real algebras. This is achieved in Theorems 15, 16, 19, 22 and 23. We advance in Section 3 a list of the equivalence classes. The center of the algebra plays a key role; this is the reason to distinguish between the cases of $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ and the case of $M_n(\mathbb{C})$.

The homogeneous components of these gradings have the same dimension, which is restricted to 1, 2 or 4 (see Section 2). The former case is analyzed in Section 5. The situation is very similar to that of the complex field, but the isomorphism classes are in correspondence with quadratic forms over the field of two elements, instead of alternating bicharacters, because of the lack of roots of unity in the real numbers. Section 4 is devoted to quadratic forms. As a curiosity, we obtain an alternative proof that the Arf invariant over the field of two elements is well defined, working with algebras in characteristic 0.

Finally in Section 6 we study the remaining case of homogeneous components of dimension 2 or 4. We have two tools to reduce the problem to the previous case. The first is Proposition 20, which states that these gradings are not fine. The second is the Double Centralizer Theorem.

In the recent preprint [4] (simultaneous to our preprint [8]), a classification up to equivalence of the division gradings on finite-dimensional simple real algebras has been considered. We solve, using different methods, the problem of classification both up to equivalence and up to isomorphism.

2. Background on gradings

In this section we review, following [5], the basic definitions and properties of gradings that will be used in the rest of the paper.

Definition 1. Let $\mathcal{D}$ be an algebra over a field $\mathbb{F}$, and let G be a group. A *G-grading* Γ on $\mathcal{D}$ is a decomposition of $\mathcal{D}$ into a direct sum of subspaces indexed by G: