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Periodicity in bilinear lattices and the Coxeter formalism

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We introduce and study in detail so-called circulant (Coxeter-periodic) elements and circulant families in a bilinear lattice $\mathcal{K}$ as well as their dual versions, called anti-circulant. We show that they form a natural environment for a systematic explanation of certain cyclotomic factors of the Coxeter polynomial $\chi_{\mathcal{K}}$ of $\mathcal{K}$ and in consequence, of Coxeter polynomials of algebras of finite global dimension. We discuss the properties of quadratic forms induced by circulant and anti-circulant families. Moreover, we interpret the results in the language of representation theory of algebras and point out applications (facts concerning tubular families in Auslander–Reiten quivers and quadratic forms of algebras). Abstract considerations in bilinear lattices are illustrated with a collection of non-trivial examples arising from module and derived categories. The results show that techniques of linear algebra and number theory provide efficient tools for explaining various representation theoretic facts.

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1. Introduction

The Coxeter spectral properties of algebras of finite global dimension have been intensively studied for the last years by many authors, see e.g. [9,13–15,20,22–25,27,28,35], cf. the survey articles [17,21]. It appears that some module theoretic properties of an algebra Λ are (sometimes surprisingly) reflected in the shape of the Coxeter formalism: the Coxeter matrix ϕ_Λ, its spectrum, the Coxeter polynomial χ_Λ and the Euler bilinear form. However, the exact nature of this relationship still remains not well understood. We note that this Coxeter formalism appears also in different mathematical contexts (like group theory, Lie theory, knot theory, singularity theory [17,21], study of coherent sheaves [15] and algebraic graph theory [4,7,10,29,30,11,12,19]) and often provides a nice connection to representation theory of algebras.

It has been observed in several papers [23,14–16,20], cf. [17], that tubes in the Auslander–Reiten quiver of the module category $\operatorname{mod}\Lambda$ or the derived category $\mathcal{D}^b(\Lambda)$ can be a reason of certain cyclotomic factors of the Coxeter polynomial χ_Λ of an algebra Λ. In the present paper we develop a general setting for a systematic study of this phenomenon in an abstract environment of *bilinear lattices* in the sense of Lenzing [14] (see Section 2 for all the basic definitions).

We introduce a notion of a *circulant* element (that is, a Coxeter-periodic element in a bilinear lattice, see Definition 3.1) and a *circulant family* (Definition 4.1), which are a K-theoretic abstraction of a tube and an orthogonal family of tubes in an Auslander–Reiten quiver, slightly different and more general than this from [14]. However, we show on a plenty of examples, that the notion of a circulant element also embraces objects which do not belong to tubes (see Examples 3.9, 3.11 and 4.7). We discuss in detail the properties of circulant elements (resp. circulant families) in Section 3 (resp. Section 4). Among others, we show several numerical limitations on the periods of circulant elements (families) and we precisely describe their influence on cyclotomic factors of the Coxeter polynomial (Proposition 3.6, Corollary 3.7 and Theorem 4.3). Moreover, we prove that under some natural assumptions, circulant elements (families) induce semidefinite quadratic forms (Proposition 3.10 and Theorem 4.6) and provide a tool for investigating the properties of the quadratic forms associated with algebras.

As an illustration of some of the consequences of our results we mention the following stronger version of deep classical results concerning tubular families in module categories.

Theorem 1. *Let Λ be a k-algebra and $\mathcal{C} = \{\mathcal{T}_i\}_{i\in I}$ a nonempty family of pairwise Hom-orthogonal tubes in the Auslander–Reiten quiver $\Gamma(\Lambda)$ of $\operatorname{mod}\Lambda$. Assume that one of the following conditions holds:*

(*a*) $\operatorname{gl.dim}\Lambda < \infty$ *and each $\mathcal{T}_i$ is hereditary and has Hom-orthogonal mouth,*
(*b*) $\operatorname{gl.dim}\Lambda/\operatorname{ann}\mathcal{C} < \infty$ *and each $\mathcal{T}_i$ is generalized standard.*