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Fast recovery and approximation of hidden Cauchy structure

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We derive an algorithm of optimal complexity which determines whether a given matrix is a Cauchy matrix, and which exactly recovers the Cauchy points defining a Cauchy matrix from the matrix entries. Moreover, we study how to approximate a given matrix by a Cauchy matrix with a particular focus on the recovery of Cauchy points from noisy data. We derive an approximation algorithm of optimal complexity for this task, and prove approximation bounds. Numerical examples illustrate our theoretical results.

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1. Introduction

Two vectors $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$ are called *Cauchy points*, if

$$s_i - t_j \neq 0 \quad \text{for all } i, j.$$

Such Cauchy points define a *Cauchy matrix*

$$C(s,t) = [c_{ij}] := \left[\frac{1}{s_i - t_j} \right].$$

Cauchy matrices occur in numerous applications. To give just one example, let $(s_i, z_i) \in \mathbb{C} \times \mathbb{C}$ be given with pairwise distinct values $s_1, \ldots, s_n$ and let $t_1, \ldots, t_n \in \mathbb{C}$ be such that $s_i \neq t_j$ for all i, j. Then the coefficients $a = [a_1, \ldots, a_n]^T \in \mathbb{C}^n$ such that the rational function

$$r(\zeta) = \sum_{j=1}^{n} \frac{a_j}{\zeta - t_j}$$

satisfies $r(s_i) = z_i$, $i = 1, \ldots, n$, can be found by solving the linear system

$$C(s,t)a = z.$$

Note that the condition $s_i - t_j \neq 0$ for the Cauchy points appears naturally in this application (as in many others) by the requirement that the poles of the rational function $r(\zeta)$ must be distinct from the points where the (finite) values of $r(\zeta)$ are prescribed.

A Cauchy matrix satisfies the Sylvester type displacement equation

$$SC(s,t) - C(s,t)T = 1_m 1_n^T,$$

where $S := \operatorname{diag}(s) \in \mathbb{C}^{m,m}$, $T := \operatorname{diag}(t) \in \mathbb{C}^{n,n}$, and $1_m := [1, \ldots, 1]^T \in \mathbb{R}^m$. Hence the $\{S, T\}$-*displacement rank* of $C(s,t)$ is equal to 1. The concept of displacement rank was originally introduced in [1,2]; see [3, Section 12.1] for an introduction. Due to this special structure, several fast algorithms exist for performing matrix computations with $C(s,t)$. For example, an LU decomposition of $C(s,t)$ with partial pivoting can be computed in $\mathcal{O}(mn)$ operations [4] (the GKO algorithm), and matrix–vector products with $C(s,t)$ can be computed very fast [5] (the fast multipole method); see also [6] and [7, Section 3.6].

In this work we are, however, not concerned with performing computations with Cauchy matrices. Rather we study the problem of determining whether a given matrix $A \in \mathbb{C}^{m,n}$ is equal or at least “close” to a Cauchy matrix. For such matrices we derive algorithms of *optimal complexity* that compute Cauchy points $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$ with $A = C(s,t)$ when A is a Cauchy matrix, or with $A \approx C(s,t)$ when certain conditions are satisfied. We are not aware that a similar study has appeared in the literature before.