Linear Algebra Appl. 493 (2016) 281–300

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Perron spectratopes and the real nonnegative inverse eigenvalue problem

Charles R. Johnson [a], Pietro Paparella [b,*]

[a] *Department of Mathematics, College of William & Mary, Williamsburg, VA 23187-8795, USA*
[b] *Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA*

ARTICLE INFO

Article history:
Received 22 August 2015
Accepted 20 November 2015
Available online 17 December 2015
Submitted by R. Brualdi

MSC:
15A18
15B48
15A29
05B20
05E30

Keywords:
Perron spectracone
Perron spectratope
Real nonnegative inverse eigenvalue problem
Hadamard matrix
Association scheme
Relative gain array

ABSTRACT

Call an n-by-n invertible matrix S a *Perron similarity* if there is a real non-scalar diagonal matrix D such that SDS^{-1} is entrywise nonnegative. We give two characterizations of Perron similarities and study the polyhedra $\mathcal{C}(S) := \{x \in \mathbb{R}^n : SD_xS^{-1} \geq 0,\ D_x := \operatorname{diag}(x)\}$ and $\mathcal{P}(S) := \{x \in \mathcal{C}(S) : x_1 = 1\}$, which we call the *Perron spectracone* and *Perron spectratope*, respectively. The set of all normalized real spectra of diagonalizable nonnegative matrices may be covered by Perron spectratopes, so that enumerating them is of interest. The Perron spectracone and spectratope of Hadamard matrices are of particular interest and tend to have large volume. For the canonical Hadamard matrix (as well as other matrices), the Perron spectratope coincides with the convex hull of its rows.
In addition, we provide a constructive version of a result due to Fiedler [9, Theorem 2.4] for Hadamard orders, and a constructive version of the Boyle–Handelman theorem [2, Theorem 5.1] for Suleĭmanova spectra.

* Corresponding author.
E-mail addresses: crjohn@wm.edu (C.R. Johnson), pietrop@uw.edu (P. Paparella).
URL: http://faculty.washington.edu/pietrop/ (P. Paparella).

http://dx.doi.org/10.1016/j.laa.2015.11.033

1. Introduction

The *real nonnegative inverse eigenvalue problem* (RNIEP) is to determine which sets of n real numbers occur as the spectrum of an n-by-n nonnegative matrix. The RNIEP is unsolved for $n \geq 5$, and the following variations, which are also unsolved for $n \geq 5$, are relevant to this work (additional background information on the RNIEP can be found in, e.g., [5,18,20]):

- *Diagonalizable RNIEP* (D-RNIEP): Determine which sets of n real numbers occur as the spectrum of an n-by-n diagonalizable nonnegative matrix.
- *Symmetric NIEP* (SNIEP): Determine which sets of n real numbers occur as the spectrum of an n-by-n symmetric nonnegative matrix.
- *Doubly stochastic RNIEP* (DS-RNIEP): Determine which sets of n real numbers occur as the spectrum of an n-by-n doubly stochastic matrix.
- *Doubly stochastic SNIEP* (DS-SNIEP): Determine which sets of n real numbers occur as the spectrum of an n-by-n symmetric doubly stochastic matrix.

The RNIEP and the SNIEP are equivalent when $n \leq 4$ and distinct otherwise (see [13]). Notice that there is no distinction between the SNIEP and the D-SNIEP since every symmetric matrix is diagonalizable.

The set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ is said to be *realizable* if there is an n-by-n nonnegative matrix with spectrum σ. If A is a nonnegative matrix that realizes σ, then A is called a *realizing matrix* for σ. It is well-known that if σ is realizable, then

$$s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0,\ \forall\ k \in \mathbb{N} \tag{1.1}$$

$$\rho(\sigma) := \max_i |\lambda_i| \in \sigma \tag{1.2}$$

$$s_k^m(\sigma) \leq n^{m-1} s_{km}(\sigma), \forall\ k, m \in \mathbb{N} \tag{1.3}$$

Condition (1.3), known as the *J-LL condition*, was proven independently by Johnson in [12], and by Loewy and London in [17].

In this paper, we introduce several polyhedral sets whose points correspond to spectra of entrywise nonnegative matrices. In particular, given a nonsingular matrix S, we define several polytopic subsets of the polyhedral cone $\mathcal{C}(S) := \{x \in \mathbb{R}^n : SD_xS^{-1} \geq 0,\ D_x := \operatorname{diag}(x)\}$ and use them to verify the known necessary and sufficient conditions for the RNIEP and SNIEP when $n \leq 4$. For a nonsingular matrix S, we provide a necessary and sufficient condition such that $\mathcal{C}(S)$ is nontrivial. For every $n \geq 1$, we characterize $\mathcal{C}(H_n)$, where H_n is the *Walsh matrix* of order 2^n, which resolves a problem posed in [7, p. 48]. Our proof method yields a highly-structured $(2^n - 1)$-class *(commutative) association scheme* and, as a consequence, a highly structured *Bose–Mesner Algebra*. In addition, we provide a constructive version of a result due to Fiedler [9, Theorem 2.4] for