Linear Algebra and its Applications 493 (2016) 301–312

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A new inverse three spectra theorem for Jacobi matrices

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ARTICLE INFO

Article history:
Received 13 September 2014
Accepted 2 December 2015
Available online 17 December 2015
Submitted by P. Semrl

MSC:
primary 15A29
secondary 15L24, 34A55

Keywords:
Jacobi matrix
Eigenvalue
Inverse three spectra theorem

ABSTRACT

This paper deals with an inverse three spectra problem for Jacobi matrices, where the spectrum of the full $N \times N$ Jacobi matrix T is prescribed, together with the spectra of two matrices obtained from the principal submatrices of T, denoted by $T_{1,n}$ and $T_{n+1,N}$, by modifying the lower right entry of $T_{1,n}$ and the upper left entry of $T_{n+1,N}$. Here n satisfying $1 \leq n < N$ is fixed. Denote by $T^-_{1,n}$ and $T^+_{n+1,N}$ the modified principal submatrices. We give conditions for three given sets of points to be the spectra of a matrix T and of its two modified principal submatrices $T^-_{1,n}$ and $T^+_{n+1,N}$ to uniquely reconstruct the original matrix T, where the matrix T is a rank 2 perturbation of $T^-_{1,n} \oplus T^+_{n+1,N}$.

1. Introduction

This paper is concerned with the inverse eigenvalue problem of recovering the finite $N \times N$ Jacobi matrix T uniquely from three sets of eigenvalues: eigenvalues of T, eigenvalues of its leading principal submatrix of order n, but the lower right entry is modified,

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http://dx.doi.org/10.1016/j.laa.2015.12.001
0024-3795/

and eigenvalues of its lower principal submatrix of order $(N-n)$, but the upper left entry is modified. Here n satisfying $1 \le n < N$ is fixed.

Consider $N \times N$ Jacobi matrices of the form

$$T := \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & \cdot & \cdot \\ b_1 & a_2 & b_2 & 0 & \cdots & \cdot & \cdot \\ 0 & b_2 & a_3 & b_3 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & a_{N-1} & b_{N-1} \\ \cdot & \cdot & \cdot & \cdot & \cdots & b_{N-1} & a_N \end{bmatrix} \tag{1.1}$$

where the b's and a's are real numbers with $b_n < 0$ for all $n = 1, \cdots, N-1$. Under the stated assumptions, it is well known [1,7,10] that all the eigenvalues of T, denoted by $\{\lambda_m\}_{m=1}^N$, are real and simple.

An inverse three spectra problem for Jacobi matrices was first studied by Gladwell and Willms [8] motivated by restructuring fixed–free spring-mass systems. They showed that removing the n-th row and column from the matrix T and denoting the resulting submatrices by $T_{1,n-1}$ (from a_1 to a_{n-1}) and $T_{n+1,N}$ (from a_{n+1} to a_N), the matrix T can be uniquely restructured if and only if $T_{1,n-1}$ and $T_{n+1,N}$ have no eigenvalues in common. Further studies were done by many authors (see, for example, [3] and references therein; [2,4,6,12–14]). Among them, Nylen and Uhlig [13] generalized the above inverse three spectra theorem by replacing 'removing' with 'cutting'. More specifically, given

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N, \qquad \mu_1^- \le \mu_2^- \le \cdots \le \mu_n^-, \qquad \mu_1^+ \le \mu_2^+ \le \cdots \le \mu_{N-n}^+ \tag{1.2}$$

for $1 \le n < N$, then there exists a unique Jacobi matrix T and two positive constants a and b such that the eigenvalues of $T_{1,n}^-$, $T_{n+1,N}^+$ and

$$T := T_{1,n}^- \oplus T_{n+1,N}^+ + (ae_n^{[N]} - be_{n+1}^{[N]})(ae_n^{[N]} - be_{n+1}^{[N]})^T \tag{1.3}$$

are $\{\mu_i^-\}_{i=1}^n$, $\{\mu_i^+\}_{i=1}^{N-n}$ and $\{\lambda_i\}_{i=1}^N$, respectively, if and only if

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_N < \mu_N \tag{1.4}$$

with $\{\mu_i\}_{i=1}^N := \{\mu_i^-\}_{i=1}^n \cup \{\mu_i^+\}_{i=1}^{N-n}$. Here both $T_{1,n}^-$ and $T_{n+1,N}^+$ are the matrices obtained from the principal submatrices $T_{1,n}$ and $T_{n+1,N}$ of T by changing their lower right and upper left entries, respectively, and $e_n^{[N]}$ denotes the n-th column of the $N \times N$ identity matrix. It is easy to see that the matrix T is essentially a rank one perturbation of $T_{1,n}^- \oplus T_{n+1,N}^+$.

The goal of this paper is to develop a new inverse three spectra theorem as the 'interior' analogue to [13, Theorems 3 and 4] and [8]. We give conditions for three given sets of points to be eigenvalues and give a complete description of the unique Jacobi matrix