

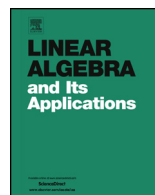


ELSEVIER

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa



The Flanders theorem over division rings



Clément de Seguins Pazzis

Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques
de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France

ARTICLE INFO

Article history:

Received 8 April 2015

Accepted 7 November 2015

Available online 18 December 2015

Submitted by P. Semrl

MSC:

15A03

15A30

Keywords:

Rank

Bounded rank spaces

Flanders's theorem

Dimension

Division ring

ABSTRACT

Let \mathbb{D} be a division ring and \mathbb{F} be a subfield of the center of \mathbb{D} over which \mathbb{D} has finite dimension d . Let n, p, r be positive integers and \mathcal{V} be an affine subspace of the \mathbb{F} -vector space $M_{n,p}(\mathbb{D})$ in which every matrix has rank less than or equal to r . Using a new method, we prove that $\dim_{\mathbb{F}} \mathcal{V} \leq \max(n, p)rd$ and we characterize the spaces for which equality holds. This extends a famous theorem of Flanders which was known only for fields.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Throughout the text, we fix a division ring \mathbb{D} , that is a non-trivial ring in which every non-zero element is invertible. We let \mathbb{F} be a subfield of the center $\mathcal{Z}(\mathbb{D})$ of \mathbb{D} and we assume that \mathbb{D} has finite dimension over \mathbb{F} .

Let n and p be non-negative integers. We denote by $M_{n,p}(\mathbb{D})$ the set of all matrices with n rows, p columns, and entries in \mathbb{D} . It has a natural structure of \mathbb{F} -vector space, which we will consider throughout the text. The kernel (or null-space) of a matrix $M \in M_{n,p}(\mathbb{D})$ is

E-mail address: dsp.prof@gmail.com.

defined as $\{X \in \mathbb{D}^p : MX = 0\}$ and denoted by $\text{Ker } M$. We denote by $E_{i,j}$ the matrix of $M_{n,p}(\mathbb{D})$ in which all the entries equal 0, except the one at the (i, j) -spot, which equals 1. The right \mathbb{D} -vector space \mathbb{D}^n is naturally identified with the space $M_{n,1}(\mathbb{D})$ of column matrices (with n rows). We naturally identify $M_{n,p}(\mathbb{D})$ with the set of all \mathbb{D} -linear maps from the right vector space \mathbb{D}^p to the right vector space \mathbb{D}^n . We have a ring structure on $M_n(\mathbb{D}) := M_{n,n}(\mathbb{D})$ with unity I_n , and its group of units is denoted by $\text{GL}_n(\mathbb{D})$.

Two matrices M and N of $M_{n,p}(\mathbb{D})$ are said to be *equivalent* when there are invertible matrices $P \in \text{GL}_n(\mathbb{D})$ and $Q \in \text{GL}_p(\mathbb{D})$ such that $N = PMQ$ (this means that M and N represent the same linear map between right vector spaces over \mathbb{D} under a different choice of bases). This relation is naturally extended to whole subsets of matrices.

The rank of a matrix $M \in M_{n,p}(\mathbb{D})$ is the rank of the family of its columns in the right \mathbb{D} -vector space \mathbb{D}^n , and it is known that it equals the rank of the family of its rows in the left \mathbb{D} -vector space $M_{1,p}(\mathbb{D})$: we denote it by $\text{rk}(M)$. Two matrices of the same size have the same rank if and only if they are equivalent.

Given a non-negative integer r , a *rank- \bar{r} subset* of $M_{n,p}(\mathbb{D})$ is a subset in which every matrix has rank less than or equal to r .

Let s and t be non-negative integers with $s \leq n$ and $t \leq p$. One defines the *compression space*

$$\mathcal{R}(s, t) := \left\{ \begin{bmatrix} A & C \\ B & [0]_{(n-s) \times (p-t)} \end{bmatrix} \mid A \in M_{s,t}(\mathbb{D}), B \in M_{n-s,t}(\mathbb{D}), C \in M_{s,p-t}(\mathbb{D}) \right\}.$$

It is obviously an \mathbb{F} -linear subspace of $M_{n,p}(\mathbb{D})$ and a $\text{rank-}\overline{s+t}$ subset. More generally, any space that is equivalent to a space of that form is called a compression space.

A classical theorem of Flanders [4] reads as follows.

Theorem 1 (*Flanders’s theorem*). *Let \mathbb{F} be a field, and n, p, r be positive integers such that $n \geq p > r$. Let \mathcal{V} be a rank- \bar{r} linear subspace of $M_{n,p}(\mathbb{F})$. Then, $\dim \mathcal{V} \leq nr$, and if equality holds then either \mathcal{V} is equivalent to $\mathcal{R}(0, r)$, or $n = p$ and \mathcal{V} is equivalent to $\mathcal{R}(r, 0)$.*

The case when $n \leq p$ can be obtained effortlessly by transposing.

Flanders’s theorem has a long history dating back to Dieudonné [3], who tackled the case when $n = p$ and $r = n - 1$ (that is, subspaces of singular matrices). Dieudonné was motivated by the study of semi-linear invertibility preservers on square matrices. Flanders came actually second [4] and, due to his use of determinants, he was only able to prove his results over fields with more than r elements (he added the restriction that the field should not be of characteristic 2, but a close examination of his proof reveals that it is unnecessary). The extension to general fields was achieved more than two decades later by Meshulam [5]. In the meantime, much progress had been made in the classification of rank- \bar{r} subspaces with dimension close to the critical one, over fields with large cardinality (see [1] for square matrices, and [2] for the generalization to rectangular

Download English Version:

<https://daneshyari.com/en/article/6416174>

Download Persian Version:

<https://daneshyari.com/article/6416174>

[Daneshyari.com](https://daneshyari.com)