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The Flanders theorem over division rings

Clément de Seguins Pazzis

Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France

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ABSTRACT

Let $\mathbb{D}$ be a division ring and $\mathbb{F}$ be a subfield of the center of $\mathbb{D}$ over which $\mathbb{D}$ has finite dimension d. Let n, p, r be positive integers and $\mathcal{V}$ be an affine subspace of the $\mathbb{F}$-vector space $\mathrm{M}_{n,p}(\mathbb{D})$ in which every matrix has rank less than or equal to r. Using a new method, we prove that $\dim_{\mathbb{F}} \mathcal{V} \leq \max(n,p)\, rd$ and we characterize the spaces for which equality holds. This extends a famous theorem of Flanders which was known only for fields.

1. Introduction

Throughout the text, we fix a division ring $\mathbb{D}$, that is a non-trivial ring in which every non-zero element is invertible. We let $\mathbb{F}$ be a subfield of the center $\mathcal{Z}(\mathbb{D})$ of $\mathbb{D}$ and we assume that $\mathbb{D}$ has finite dimension over $\mathbb{F}$.

Let n and p be non-negative integers. We denote by $\mathrm{M}_{n,p}(\mathbb{D})$ the set of all matrices with n rows, p columns, and entries in $\mathbb{D}$. It has a natural structure of $\mathbb{F}$-vector space, which we will consider throughout the text. The kernel (or null-space) of a matrix $M \in \mathrm{M}_{n,p}(\mathbb{D})$ is

E-mail address: dsp.prof@gmail.com.

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defined as $\{X \in \mathbb{D}^p : MX = 0\}$ and denoted by $\operatorname{Ker} M$. We denote by $E_{i,j}$ the matrix of $\mathrm{M}_{n,p}(\mathbb{D})$ in which all the entries equal 0, except the one at the (i, j)-spot, which equals 1. The right $\mathbb{D}$-vector space $\mathbb{D}^n$ is naturally identified with the space $\mathrm{M}_{n,1}(\mathbb{D})$ of column matrices (with n rows). We naturally identify $\mathrm{M}_{n,p}(\mathbb{D})$ with the set of all $\mathbb{D}$-linear maps from the right vector space $\mathbb{D}^p$ to the right vector space $\mathbb{D}^n$. We have a ring structure on $\mathrm{M}_n(\mathbb{D}) := \mathrm{M}_{n,n}(\mathbb{D})$ with unity I_n, and its group of units is denoted by $\mathrm{GL}_n(\mathbb{D})$.

Two matrices M and N of $\mathrm{M}_{n,p}(\mathbb{D})$ are said to be *equivalent* when there are invertible matrices $P \in \mathrm{GL}_n(\mathbb{D})$ and $Q \in \mathrm{GL}_p(\mathbb{D})$ such that $N = PMQ$ (this means that M and N represent the same linear map between right vector spaces over $\mathbb{D}$ under a different choice of bases). This relation is naturally extended to whole subsets of matrices.

The rank of a matrix $M \in \mathrm{M}_{n,p}(\mathbb{D})$ is the rank of the family of its columns in the right $\mathbb{D}$-vector space $\mathbb{D}^n$, and it is known that it equals the rank of the family of its rows in the left $\mathbb{D}$-vector space $\mathrm{M}_{1,p}(\mathbb{D})$: we denote it by $\operatorname{rk}(M)$. Two matrices of the same size have the same rank if and only if they are equivalent.

Given a non-negative integer r, a *rank-$\overline{r}$ subset* of $\mathrm{M}_{n,p}(\mathbb{D})$ is a subset in which every matrix has rank less than or equal to r.

Let s and t be non-negative integers with $s \leq n$ and $t \leq p$. One defines the *compression space*

$$\mathcal{R}(s,t) := \left\{ \begin{bmatrix} A & C \\ B & [0]_{(n-s)\times(p-t)} \end{bmatrix} \mid A \in \mathrm{M}_{s,t}(\mathbb{D}),\ B \in \mathrm{M}_{n-s,t}(\mathbb{D}),\ C \in \mathrm{M}_{s,p-t}(\mathbb{D}) \right\}.$$

It is obviously an $\mathbb{F}$-linear subspace of $\mathrm{M}_{n,p}(\mathbb{D})$ and a rank-$\overline{s+t}$ subset. More generally, any space that is equivalent to a space of that form is called a compression space.

A classical theorem of Flanders [4] reads as follows.

Theorem 1 *(Flanders's theorem). Let $\mathbb{F}$ be a field, and n, p, r be positive integers such that $n \geq p > r$. Let $\mathcal{V}$ be a rank-$\overline{r}$ linear subspace of $\mathrm{M}_{n,p}(\mathbb{F})$. Then, $\dim \mathcal{V} \leq nr$, and if equality holds then either $\mathcal{V}$ is equivalent to $\mathcal{R}(0,r)$, or $n = p$ and $\mathcal{V}$ is equivalent to $\mathcal{R}(r,0)$.*

The case when $n \leq p$ can be obtained effortlessly by transposing.

Flanders's theorem has a long history dating back to Dieudonné [3], who tackled the case when $n = p$ and $r = n - 1$ (that is, subspaces of singular matrices). Dieudonné was motivated by the study of semi-linear invertibility preservers on square matrices. Flanders came actually second [4] and, due to his use of determinants, he was only able to prove his results over fields with more than r elements (he added the restriction that the field should not be of characteristic 2, but a close examination of his proof reveals that it is unnecessary). The extension to general fields was achieved more than two decades later by Meshulam [5]. In the meantime, much progress had been made in the classification of rank-$\overline{r}$ subspaces with dimension close to the critical one, over fields with large cardinality (see [1] for square matrices, and [2] for the generalization to rectangular