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# Local derivations on finite-dimensional Lie algebras

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ABSTRACT

We prove that every local derivation on a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero is a derivation. We also give examples of finite-dimensional nilpotent Lie algebras $\mathcal{L}$ with $\dim \mathcal{L} \geq 3$ which admit local derivations which are not derivations.



## 1. Introduction

In 1990, Kadison [11] and Larson and Sourour [12] introduce the following concept of local derivation: Let $X$ be a Banach $A$-bimodule over a Banach algebra $A$, a linear

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mapping $\Delta : A \to X$ is said to be a *local derivation* if for every $x$ in $A$ there exists a derivation $D_x : A \to X$, depending on $x$, satisfying $\Delta(x) = D_x(x)$.

The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations [5,11,12]. Kadison proves in [11, Theorem A] that each continuous local derivation of a von Neumann algebra $M$ into a dual Banach $M$-bimodule is a derivation. This theorem gave rise to studies and several results on local derivations on C$^*$-algebras, culminating with a definitive contribution due to Johnson, which asserts that every continuous local derivation of a C$^*$-algebra $A$ into a Banach $A$-bimodule is a derivation [10, Theorem 5.3]. Moreover in his paper, Johnson also gives an automatic continuity result by proving that local derivations of a C$^*$-algebra $A$ into a Banach $A$-bimodule $X$ are continuous even if not assumed a priori to be so (cf. [10, Theorem 7.5]).

Investigation of local derivations on (not necessarily Banach) algebras of unbounded operators were initiated in papers [1] and [2]. A comprehensive survey of recent results concerning local and 2-local derivations on C$^*$- and von Neumann algebras is presented in [4].

The paper [1] is devoted to the study of local derivations on the algebra $S(M, \tau)$ of all $\tau$-measurable operators affiliated with a von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$. One of the main results in the mentioned paper presents an unbounded version of Kadison's Theorem A from [11] and it asserts that every local derivation on $S(M, \tau)$ which is continuous in the measure topology automatically becomes a derivation. In particular in the case of the type I von Neumann algebra $M$ all such local derivations on $S(M, \tau)$ are inner derivations. Moreover for type I finite von Neumann algebras without abelian direct summands as well as for von Neumann algebras with the atomic lattice of projections, the continuity condition on local derivations is redundant. In [2] it was proved that each local derivation on the so-called non-commutative Arens algebras affiliated with a von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$ is automatically a derivation.

The paper [1] also deals with the problem of existence of local derivations which are not derivations on algebras of measurable operators. The consideration of such examples on various finite- and infinite-dimensional algebras was initiated by Kadison, Kaplansky and Jensen (see [11]). In [1] this problem has been solved for a class of commutative regular algebras, which include the algebras of measurable functions on a measure space. Namely necessary and sufficient conditions were obtained for the algebras of measurable and $\tau$-measurable operators affiliated with a commutative von Neumann algebra to admit local derivations that are not derivations.

In [3] we initiated the study of derivation type maps on non-associative algebras, namely, we investigated so-called 2-local derivations on finite-dimensional Lie algebras, and showed an essential difference between semisimple and nilpotent Lie algebras in the behavior of their 2-local derivations. Recently Z. Chen and D. Wang [6] have proved that any 2-local automorphism on a simple Lie algebra of type $A_l$, $D_l$, $E_k$ $(k = 6, 7, 8)$ over an algebraically closed field of characteristic zero is an automorphism.