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Nonlinear maps preserving the reduced minimum modulus of operators ☆

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ABSTRACT

Let X and Y be infinite-dimensional complex Banach spaces, and let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y). We describe bijective bicontinuous maps φ from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ satisfying

$$\gamma(\varphi(S \pm \varphi(T))) = \gamma(S \pm T)$$

for all $S,\ T \in \mathscr{B}(X)$, where $\gamma(T)$ is the reduced minimum modulus of an operator T. An analogue result for the finite-dimensional case is obtained.

1. Introduction

Numerous studies have been done on the subject of nonlinear preserver problems. These problems, in the most general setting, demand the characterization of maps be-

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tween algebras that leave a certain property, a particular relation, or even a subset invariant without assuming in advance algebraic conditions such as linearity, additivity or multiplicity; see for instance [8,12,13,16,15,14,17,18,22,20,21,19,23–32]. The characterization of surjective maps on the algebra $M_n(\mathbb{C})$ of all complex $n \times n$-matrices preserving the spectral radius of the difference of matrices was given in [1] by Bhatia, Šemrl and Sourour. In [28], Molnár studied maps preserving the spectrum of matrix or Hilbert space operator products. His results have been extended in many directions for uniform algebras and semisimple commutative Banach algebras, and a number of results is obtained on maps preserving several spectral and local spectral quantities of operator or matrix product, or Jordan product, or Jordan triple product, or difference; see for instance [3–6,8–11,13–17,19,21–27,29–31] and the references therein.

Recently, Bourhim, Mashreghi and Stepanyan described in [7] nonlinear maps preserving the minimum and surjectivity moduli of the difference of operators and matrices, and thus extending the main results of several papers to the nonlinear setting; see for instance [2] and the references therein. However, the corresponding problem of characterizing nonlinear maps preserving the reduced minimum modulus was naturally left therein [7]. It is the aim of this note to describe such maps and show that a bijective bicontinuous map on the algebra of all bounded linear operators on a complex Banach space preserves the reduced minimum modulus of the differences of operators if and only if it is an isometry translated by an operator. The proof of such a promised result uses some arguments that are influenced by ideas from several papers including [2,7].

2. Preliminaries

Let $\mathscr{M}_n(\mathbb{C})$ denote, as usual, the algebra of all $n \times n$ complex matrices, and let T^{tr} denote the transpose of any matrix $T \in \mathscr{M}_n(\mathbb{C})$. Let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) be the algebra of all bounded linear operators on a complex Banach space X (resp. Y). The dual space of X is denoted by X^*, and the Banach space adjoint of an operator $T \in \mathscr{B}(X)$ is denoted by T^*. The *minimum modulus* of an operator $T \in \mathscr{B}(X)$ is $\mathrm{m}(T) := \inf\{\, \|Tx\| \colon x \in X, \|x\| = 1\,\}$, and is positive precisely when T is bounded below; i.e., T is injective and has a closed range. The *surjectivity modulus* of T is $\mathrm{q}(T) := \sup\{\, \varepsilon \geq 0 \colon \varepsilon B_X \subseteq T(B_X)\,\}$, and is positive if and only if T is surjective. Here, B_X is the closed unit ball of X. While, the *maximum modulus* of T is defined by $\mathrm{M}(T) := \max\{\mathrm{m}(T), \mathrm{q}(T)\}$, and is positive precisely when either T is bounded below or T is surjective. Note that $\mathrm{m}(T^*) = \mathrm{q}(T)$ and $\mathrm{q}(T^*) = \mathrm{m}(T)$ for all $T \in \mathscr{B}(X)$, and consequently $\mathrm{M}(T^*) = M(T)$ for all $T \in \mathscr{B}(X)$. Finally, recall that the reduced minimum modulus of T is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \ \mathrm{dist}(x, \operatorname{Ker} T) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0 \end{cases}$$

and is positive if and only if the range of T is closed. It is easy to see that $\gamma(T^*) = \gamma(T)$ and $\gamma(T) \geq \mathrm{M}(T)$. Moreover, we note that $\gamma(T) = \mathrm{M}(T)$ if $\mathrm{M}(T) > 0$, and