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The smallest eigenvalues of the 1-point fixing graph

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Let $\mathcal{S}_n$ be the symmetric group on $\{1, \ldots, n\}$. The k-point fixing graph $\mathcal{F}(n, k)$ is defined to be the graph with vertex set $\mathcal{S}_n$ and two vertices g, h of $\mathcal{F}(n, k)$ are joined if and only if gh^{-1} fixes exactly k points. In this paper, we determine the smallest eigenvalue for $\mathcal{F}(n, 1)$.

1. Introduction

Let G be a finite group and S be an inverse closed subset of G, i.e., $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in S$. The *Cayley graph* $\Gamma(G, S)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = su$ for some $s \in S$.

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A Cayley graph $\Gamma(G, S)$ is said to be *normal* if S is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of G.

Theorem 1.1. *(See [1,5,18,19].) The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$ and $S \subseteq \mathcal{S}_n$ be closed under conjugation. Since central characters are algebraic integers [12, Theorem 3.7 on p. 36] and that the characters of the symmetric group are integers ([12, 2.12 on p. 31] or [22, Corollary 2 on p. 103]), by Theorem 1.1, the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ are integers.

Corollary 1.2. *The eigenvalues of a normal Cayley graph $\Gamma(\mathcal{S}_n, S)$ are integers.*

For $k \leq n$, a *k-permutation* of $[n]$ is an injective function from $[k]$ to $[n]$. So any k-permutation π can be represented by a vector $(i_1, \ldots, i_k)$ where $\pi(j) = i_j$ for $j = 1, \ldots, k$. Let $1 \leq r \leq k \leq n$. The *(n, k, r)-arrangement graph* $A(n, k, r)$ has all the k-permutations of $[n]$ as vertices and two k-permutations are adjacent if they differ in exactly r positions. Formally, the vertex set $V(n, k)$ and edge set $E(n, k, r)$ of $A(n, k, r)$ are

$$\begin{aligned} V(n,k) &= \big\{(p_1, p_2, \ldots, p_k) \mid p_i \in [n] \text{ and } p_i \neq p_j \text{ for } i \neq j\big\}, \\ E(n,k,r) &= \big\{\{(p_1, p_2, \ldots, p_k), (q_1, q_2, \ldots, q_k)\} \mid p_i \neq q_i \text{ for } i \in R \text{ and} \\ &\qquad p_j = q_j \text{ for all } j \in [k] \setminus R \text{ for some } R \subseteq [k] \text{ with } |R| = r\big\}. \end{aligned}$$

Note that $|V(n, k)| = n!/(n - k)!$ and $A(n, k, r)$ is a regular graph [3, Theorem 4.2]. In particular, $A(n, k, 1)$ is a $k(n - k)$-regular graph. We note here that $A(n, k, 1)$ was first introduced in [4] as an interconnection network model for parallel computation. Furthermore, $A(n, k, 1)$ is called the partial permutation graph by Krakovski and Mohar in [13]. The eigenvalues of the arrangement graphs $A(n, k, 1)$ were first studied in [2] by using a method developed by Godsil and McKay [10]. A relation between the eigenvalues of $A(n, k, r)$ and certain Cayley graphs was given in [3].

The *derangement graph* Γ_n is the Cayley graph $\Gamma(\mathcal{S}_n, D_n)$ where D_n is the set of derangements in $\mathcal{S}_n$. That is, two vertices g, h of Γ_n are joined if and only if $g(i) \neq h(i)$ for all $i \in [n]$, or equivalently gh^{-1} fixes no point. Since D_n is closed under conjugation, by Corollary 1.2, the eigenvalues of the derangement graph are integers. The lower and