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On the number of integral graphs

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ABSTRACT

We show that at most a $2^{-cn^{3/2}}$ proportion of graphs on n vertices have integral spectrum. This improves on previous results of Ahmadi, Alon, Blake, and Shparlinski (2009), who showed that the proportion of such graphs was exponentially small.

1. Introduction and statement of main results

Call a graph **Integral** if all eigenvalues of its adjacency matrix are integers. Examples of integral graphs include the hypercube, the complete graph K_n, the symmetric complete bipartite graph $K_{n,n}$, and the Paley graph on q vertices where q is an odd square prime power.

Integral graphs were first studied by Harary and Schwenk [6], who gave several families of integral graphs, but at the same time described the general question of classifying all

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integral graphs as "intractable". More recently, there have been several papers noting that integral graphs may be useful in designing quantum spin networks with perfect state transfer (see, for instance, [3,2]).

It is natural to try and count these graphs, or, equivalently, to give the probability that a random graph (one chosen uniformly at random from the $2^{\binom{n}{2}}$ graphs on n vertices) is integral. The first non-trivial upper bound on this problem was given by Ahmadi, Alon, Blake, and Shparlinski [1]. In probabilistic language, the bound they obtained was

Theorem 1. *The probability that a randomly chosen graph on n vertices is integral is, for sufficiently large n, at most $2^{-n/400}$,*

and they noted in their paper that "we believe our bound is far from being tight and the number of integral graphs is substantially smaller". Our main result confirms this belief, showing that the proportion of integral graphs decays much faster than exponentially.

Theorem 2. *The probability that a randomly chosen graph on n vertices is integral is, for large n, at most $2^{-cn^{3/2}}$, for some absolute constant c.*

Remark 1. This result is likely still not close to being tight. For more on this, see the final section of this paper.

In the next section we will collect a few linear algebraic properties that hold for the adjacency matrix of every graph, deterministic or random. The proof of Theorem 2 will follow from combining these properties with a counting argument, essentially tracking how the spectrum of the adjacency matrix behaves as the graph grows.

2. A few deterministic observations

We begin with the quick observation that adjacency matrices of graphs cannot have too many large eigenvalues.

Lemma 1. *Let G be an arbitrary graph on n vertices, and let A be the adjacency matrix of G. Then A must contain at least $\frac{3n}{4}$ eigenvalues in the interval $[-2\sqrt{n}, 2\sqrt{n}]$.*

Remark 2. It follows from Wigner's semicircular law [12] together with interlacing that for large n almost every graph on n vertices has fewer than $\frac{3n}{4}$ eigenvalues in the interval $[-1.26\sqrt{n}, 1.26\sqrt{n}]$.

Proof. Let λ_i be the i-th eigenvalue of A. We have

$$\sum_{i=1}^{n} \lambda_i^2 = Tr(A^2) = 2E(G) \leq n^2,$$