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An S-type eigenvalue localization set for tensors

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ABSTRACT

An S-type eigenvalue localization set for a tensor is given by breaking $N = \{1, 2, \cdots, n\}$ into disjoint subsets S and its complement. It is shown that the new set is tighter than those provided by Qi (2005) [21] and Li et al. (2014) [13]. As applications of the results, a checkable sufficient condition for the positive definiteness of tensors and a checkable sufficient condition of the positive semi-definiteness of tensors are given.

1. Introduction

Eigenvalue problems of tensors have become an important topic of study in numerical multilinear algebra, and they have a wide range of practical applications; see [2–5, 7–11,18–20,22,23,25,26]. Here we call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order m dimension n, denoted by $\mathcal{A} \in C^{[m,n]}$ $(R^{[m,n]})$, if

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$$a_{i_1\cdots i_m} \in C\ (R),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ associated with λ [17,21], where $\mathcal{A}x^{m-1}$ is an n-dimensional vector whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}\ (N = \{1, 2, \ldots, n\})$$

and

$$x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T.$$

If x and λ are all real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ [21,22,28].

One of many practical applications of eigenvalues of tensors is that one can identify the positive (semi-)definiteness for an even-order real symmetric tensor by using the smallest H-eigenvalue of a tensor, consequently, can identify the positive (semi-)definiteness of the multivariate homogeneous polynomial determined by this tensor, for details, see [11,21].

Because it is not easy to compute the smallest H-eigenvalue of tensors when the order and dimension are large, one always tries to give a set including all eigenvalues in the complex plane [13–16,21]. In particular, if this set for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest H-eigenvalue is positive, consequently, the corresponding tensor is positive definite.

In [21], Qi gave an eigenvalue localization set for real symmetric tensors, which is a generalization of the well-known Geršgorin's eigenvalue localization set of matrices [6,24]. This result can be easily generalized to general tensors [13,27].

Theorem 1. *(See [13,21,27].) Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in C^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) := \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$,

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\cdots i}| \leq r_i(\mathcal{A})\},\ r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N,\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|$$