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Eigenvalue multiplicity in triangle-free graphs

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ABSTRACT

Let G be a connected triangle-free graph of order $n > 5$ with $\mu \notin \{-1, 0\}$ as an eigenvalue of multiplicity $k > 1$. We show that if d is the maximum degree in G then $k \leq n - d - 1$; moreover, if $k = n - d - 1$ then either (a) G is non-bipartite and $k \leq (\mu^2 + 3\mu + 1)(\mu^2 + 2\mu - 1)$, with equality only if G is strongly regular, or (b) G is bipartite and $k \leq d - 1$, with equality only if G is a bipolar cone. In each case we discuss the extremal graphs that arise.

1. Introduction

Let G be a graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus if G has $(0, 1)$-adjacency matrix A then the eigenspace $\mathcal{E}_A(\mu)$ has dimension k and codimension t. From [1, Theorem 3.1], we know that if $\mu \notin \{-1, 0\}$ and $t > 2$ then $k \leq n - \frac{1}{2}(-1 + \sqrt{8n + 1})$, equivalently $k \leq \frac{1}{2}t(t - 1)$. This bound, which is sharp for $t = 8$, has been improved for several classes of graphs, such as regular graphs [1], cubic graphs [12], trees [10], and graphs with prescribed girth [11]. For each class, it is of

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interest to describe the graphs for which a sharp bound is attained. Here we investigate the situation in which G is a connected triangle-free graph with maximum degree d. We show first that if $\mu \notin \{-1, 0\}$ and G is not a star then $k \leq n - d - 1$: this bound (which is immediate from interlacing when $\mu^2 \neq d$) is an improvement on the general bound when $d > \frac{1}{2}(-3 + \sqrt{8n+1})$. Next we prove that when $k = n - 1 - d$, equivalently $t = d + 1$, the following hold: (i) G has the star $K_{1,d}$ as a star complement for μ, (ii) if G is non-bipartite of order $n > 5$ then $d = \mu(\mu^2 + 3\mu + 1)$ and $k \leq (\mu^2 + 3\mu + 1)(\mu^2 + 2\mu - 1)$, with equality if and only if $\mu \in \mathbb{N}$ and G is strongly regular with parameters $(\mu^2 + 3\mu)^2$, $\mu(\mu^2 + 3\mu + 1)$, 0, $\mu(\mu + 1)$, (iii) if G is bipartite then $k \leq d - 1$. The idea of the proof is to show that when k is as large as possible, G is regular or bipartite, so that we may apply the results of [14] or [13] respectively. It follows that in both cases there is a close relation between symmetric 2-designs and the extremal graphs that arise. The bipartite graphs for which $n - 1 - d = k = d - 1$ are discussed further in Section 4.

We write $G = SRG(n, r, e, f)$ to mean that G is strongly regular with parameters n, r, e, f. Note that if $G = SRG((\mu^2 + 3\mu)^2, \mu(\mu^2 + 3\mu + 1), 0, \mu(\mu + 1))$ then any induced subgraph of G containing $K_{1,d}$ is a triangle-free graph satisfying the condition $t = d + 1$ in respect of the eigenvalue μ. The specific graphs cited in Section 3 show that not all examples arise in this way, even when $d = \mu(\mu^2 + 3\mu + 1)$ and μ is taken to be a non-main eigenvalue (that is, an eigenvalue for which $\mathcal{E}_A(\mu)$ is orthogonal to the all-1 vector in $\mathbb{R}^n$).

2. Preliminaries

Let G be a graph of order n with μ as an eigenvalue of multiplicity k. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [3, Chapter 5]. We shall require the following results, where for any $X \subseteq V(G)$, we write G_X for the subgraph of G induced by X. We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. Further notation may be found in the monograph [3].

Theorem 2.1. *(See [3, Theorem 5.1.7].) Let X be a set of vertices in the graph G. Suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top (\mu I - C)^{-1} B. \tag{1}$$

With the notation of Theorem 2.1, let X be a star set for μ in G, and let $H = G - X$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n-k}$, we write $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \mathbf{x}^\top (\mu I - C)^{-1} \mathbf{y}$. The columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$. Eq. (1) shows that