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On families of anticommuting matrices

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ABSTRACT

Let $e_1,\dots,e_k$ be complex $n\times n$ matrices such that $e_ie_j=-e_je_i$ whenever $i\neq j$. We conjecture that $\mathrm{rk}(e_1^2)+\mathrm{rk}(e_2^2)+\dots+\mathrm{rk}(e_k^2)\leq O(n\log n)$. We show that:

(i). $\mathrm{rk}(e_1^n)+\mathrm{rk}(e_2^n)+\dots+\mathrm{rk}(e_k^n)\leq O(n\log n)$,
(ii). if $e_1^2,\dots,e_k^2\neq 0$ then $k\leq O(n)$,
(iii). if $e_1,\dots,e_k$ have full rank, or at least $n-O(n/\log n)$, then $k\leq O(\log n)$.

(i) implies that the conjecture holds if $e_1^2,\dots,e_k^2$ are diagonalisable (or if $e_1,\dots,e_k$ are). (ii) and (iii) show it holds when their rank is sufficiently large or sufficiently small.

1. Introduction

Consider a family $e_1,\dots,e_k$ of complex $n\times n$ matrices which pairwise anticommute; i.e., $e_ie_j=-e_je_i$ whenever $i\neq j$. A standard example is a representation of a Clifford algebra, which gives an anticommuting family of $2\log_2 n+1$ invertible matrices, if n is a power of two (see Example 1 in Section 3). This is known to be tight: if all the matrices $e_1,\dots,e_k$ are invertible then k is at most $2\log_2 n+1$. (see [10] and Theorem 1 below).

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However, the situation is much less understood when the matrices are singular. As an example, consider the following problem:

Question 1. *Assume that every e_i has rank at least $2n/3$. Is k at most $O(\log n)$?*

We expect the answer to be positive. However, we can solve such a problem only under some extra assumptions. In [6], it was shown that an anticommuting family of diagonalisable matrices can be "decomposed" into representations of Clifford algebras. This indeed affirmatively answers Question 1 if the e_i's are diagonalisable. In this paper, we formulate a conjecture which relates the size of an anticommuting family with the rank of matrices in the family. We prove some partial results in this direction. In Theorem 3, we show that the situation is clear when the matrices are diagonalisable, or their squares are diagonalisable, or even $\mathrm{rk}(e_i^2) = \mathrm{rk}(e_i^3)$. However, we can say very little about the case when the matrices are nilpotent. In Theorem 2, we show that, in Question 1, we have $k \leq O(n)$. Theorem 6 implies that $k \leq O(\log n)$ whenever the rank of every e_i is almost full.

One motivation for this study is to understand sum-of-squares composition formulas. A sum-of-squares formula is an identity

$$(x_1^2 + x_2^2 + \cdots + x_k^2) \cdot (y_1^2 + y_2^2 + \cdots + y_k^2) = f_1^2 + f_2^2 + \cdots + f_n^2\,, \tag{1}$$

where $f_1, \ldots, f_n$ are bilinear complex[1] polynomials. We want to know how large must n be in terms of k so that such an identity exists. This problem has a very interesting history, and we refer the reader to the monograph [10] for details. A classical result of Hurwitz [3] states that $n = k$ can be achieved only for $k \in \{1, 2, 4, 8\}$. Hence, n is strictly larger than k for most values of k, but it is not known how much larger. In particular, we do not known whether[2] $n = \Omega(k^{1+\epsilon})$ for some $\epsilon > 0$. In [1], it was shown that such a lower bound would resolve an open problem in arithmetic complexity theory (while the authors obtained an $\Omega(n^{7/6})$ lower bound on *integer* composition formulas in [2]). We point out that our conjecture about anticommuting families implies $n = \Omega(k^2/\log k)$, which would be asymptotically tight. This connection is hardly surprising: already Hurwitz's theorem, as well as the more general Hurwitz–Radon theorem [4,9], can be proved by reduction to an anticommuting system.

2. Preliminaries

A family $e_1, \ldots, e_k$ of $n \times n$ complex matrices will be called *anticommuting* if $e_i e_j = -e_j e_i$ holds for every *distinct* $i, j \in \{1, \ldots, k\}$. We conjecture that the following holds ($\mathrm{rk}(A)$ is the rank of the matrix A):

[1] The problem is often phrased over $\mathbb{R}$ when the bilinearity condition is automatic.

[2] Recall that $f(k) = \Omega(g(k))$ if there exists $c > 0$ such that $f(k) \geq cg(k)$ holds for every sufficiently large k.