Linear Algebra and its Applications 493 (2016) 508–536

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Upper triangular matrices and Billiard Arrays

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ARTICLE INFO

Article history:
Received 12 September 2015
Accepted 21 December 2015
Available online 4 January 2016
Submitted by R. Brualdi

MSC:
primary 05E15
secondary 15A21

Keywords:
Upper triangular matrix
Billiard Array
Flag
Quantum group
Equitable presentation

ABSTRACT

Fix a nonnegative integer d, a field $\mathbb{F}$, and a vector space V over $\mathbb{F}$ with dimension $d+1$. Let T denote an invertible upper triangular matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$. Using T we construct three flags on V. We find a necessary and sufficient condition on T for these three flags to be totally opposite. In this case, we use these three totally opposite flags to construct a Billiard Array B on V. It is known that B is determined up to isomorphism by a certain triangular array of scalar parameters called the B-values. We compute these B-values in terms of the entries of T. We describe the set of isomorphism classes of Billiard Arrays in terms of upper triangular matrices.

1. Introduction

This paper is about a connection between upper triangular matrices and Billiard Arrays. The Billiard Array concept was introduced in [15]. This concept is closely related to the equitable presentation of $U_q(\mathfrak{sl}_2)$ [10,15]. For more information about the equitable presentation, see [1,3,5–9,12–14,16].

We now summarize our results. Fix a nonnegative integer d, a field $\mathbb{F}$, and a vector space V over $\mathbb{F}$ with dimension $d+1$. Let T denote an invertible upper triangular matrix

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http://dx.doi.org/10.1016/j.laa.2015.12.023
0024-3795/

in $\mathrm{Mat}_{d+1}(\mathbb{F})$. View T as the transition matrix from a basis $\{u_i\}_{i=0}^d$ of V to a basis $\{v_i\}_{i=0}^d$ of V. Using T we construct three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ on V as follows. For $0 \le i \le d$,

$$U_i = \mathbb{F}u_0 + \mathbb{F}u_1 + \cdots + \mathbb{F}u_i = \mathbb{F}v_0 + \mathbb{F}v_1 + \cdots + \mathbb{F}v_i;$$
$$U'_i = \mathbb{F}u_d + \mathbb{F}u_{d-1} + \cdots + \mathbb{F}u_{d-i};$$
$$U''_i = \mathbb{F}v_d + \mathbb{F}v_{d-1} + \cdots + \mathbb{F}v_{d-i}.$$

In our first main result, we find a necessary and sufficient condition (called very good) on T for $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ to be totally opposite in the sense of [15, Definition 12.1].

In [15, Theorem 12.7] it is shown how three totally opposite flags on V correspond to a Billiard Array on V. Assume that the three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ are totally opposite, and let B denote the corresponding Billiard Array on V. By [15, Lemma 19.1] B is determined up to isomorphism by a certain triangular array of scalar parameters called the B-values. In our second main result, we compute these B-values in terms of the entries of T.

Let $\mathcal{T}_d(\mathbb{F})$ denote the set of very good upper triangular matrices in $\mathrm{Mat}_{d+1}(\mathbb{F})$. Define an equivalence relation $\sim$ on $\mathcal{T}_d(\mathbb{F})$ as follows. For $T, T' \in \mathcal{T}_d(\mathbb{F})$, we declare $T \sim T'$ whenever there exist invertible diagonal matrices $H, K \in \mathrm{Mat}_{d+1}(\mathbb{F})$ such that $T' = HTK$. In our third main result, we display a bijection between the following two sets:

(i) the equivalence classes for $\sim$ on $\mathcal{T}_d(\mathbb{F})$;
(ii) the isomorphism classes of Billiard Arrays on V.

We give a commutative diagram that illustrates our second and third main results. At the end of this paper, we give an example. In this example, we display a very good upper triangular matrix with entries given by q-binomial coefficients. We show that for the corresponding Billiard Array B, all the B-values are equal to q^{-1}.

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 contains necessary facts about decompositions and flags. Section 4 is devoted to the correspondence between very good upper triangular matrices and totally opposite flags. This section contains our first main result. Section 5 contains necessary facts about Billiard Arrays. In Sections 6–8 we obtain our second and third main results. In Section 9, we display an example to illustrate our theory.

2. Preliminaries

Throughout the paper, we fix the following notation. Let $\mathbb{R}$ denote the field of real numbers. Recall the ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Fix $d \in \mathbb{N}$. Let $\{x_i\}_{i=0}^d$ denote a sequence. We call x_i the *i-component* of