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# Contour integral solutions of Sylvester-type matrix equations

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ABSTRACT

The linear matrix equations $AXB - CXD = E$, $AX - X^*D = E$, and $AXB - X^* = E$ are studied. In the case of uniqueness the solutions are expressed in terms of contour integrals.



## 1. Introduction

In this note we use contour integrals to study the matrix equations

$$AXB - CXD = E \tag{1}$$

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and

$$AX - X^*D = E \tag{2}$$

and

$$AXB - X^* = E. \tag{3}$$

We assume $A, C \in \mathbb{C}^{m\times m}$ and $B, D \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{m\times n}$, and $m = n$ in (2) and (3). The generalized Sylvester equation (1) appears in sensitivity problems of eigenvalues [12], in numerical solutions of implicit ordinary differential equations [6], and in a Lyapunov theory of descriptor systems [1]. Equation (2) – where $X^*$ denotes the conjugate transpose of $X$ – plays a role in palindromic eigenvalue problems [2,9]. The main reference to (3) is [13]. Besides (2) and (3) equations of the form $AX - X^TD = E$ and $X - CX^TD = E$ – where $X^T$ is the transpose of $X$ – have been studied e.g. in [5,13,3].

Conditions for uniqueness of solutions of (1)–(3) involve matrix pencils and their spectra. Let the pencil $zC - A$ be regular, that is $\det(\lambda C - A) \neq 0$ for some $\lambda \in \mathbb{C}$. We set

$$\sigma(zC - A) = \{\lambda;\ \det(\lambda C - A) = 0\} \cup \{\lambda;\ \det(C - \lambda^{-1}A) = 0\}.$$

Then $0 \in \sigma(zC - A)$ is equivalent to $\det A = 0$. By convention $1/\infty = 0$ and $1/0 = \infty$. Thus $\infty \in \sigma(zC - A)$ if and only if $\det C = 0$. A *contour* in the complex plane will always mean a positively oriented simple closed curve.

## 2. Explicit solutions by contour integrals

### *2.1. The generalized Sylvester equation*

It is known (see [4]) that equation (1) has a unique solution if and only if the matrix pencils

$$zC - A \text{ and } zB - D \text{ are regular,} \tag{4}$$

and

$$\sigma(zC - A) \cap \sigma(zB - D) = \emptyset. \tag{5}$$

In the following we give an explicit description of the unique solution of (1).

**Theorem 2.1.** *If the conditions (4) and (5) are satisfied then there exists a contour $\gamma$ such that $\sigma(zC - A)$ is in the interior of $\gamma$ and $\sigma(zB - D)$ is outside of $\gamma$. The unique solution of (1) is given by*

$$X = \frac{1}{2\pi i} \oint_\gamma (zC - A)^{-1} E (zB - D)^{-1} dz. \tag{6}$$