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On low-rank approximability of solutions to high-dimensional operator equations and eigenvalue problems

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ABSTRACT

Low-rank tensor approximation techniques attempt to mitigate the overwhelming complexity of linear algebra tasks arising from high-dimensional applications. In this work, we study the low-rank approximability of solutions to linear systems and eigenvalue problems on Hilbert spaces. Although this question is central to the success of all existing solvers based on low-rank tensor techniques, very few of the results available so far allow to draw meaningful conclusions for higher dimensions. In this work, we develop a constructive framework to study low-rank approximability. One major assumption is that the involved linear operator admits a low-rank representation with respect to the chosen tensor format, a property that is known to hold in a number of applications. Simple conditions, which are shown to hold for a fairly general problem class, guarantee that our derived low-rank truncation error estimates do not deteriorate as the dimensionality increases.

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1. Introduction

The past few years have seen a growing activity in applying low-rank tensor techniques to the approximate solution of high-dimensional problems, see, e.g., [11,13] for survey. The success of these techniques crucially depends on the ability to approximate the object of interest by a tensor of low rank with respect to the chosen tensor format. Although this property has been frequently confirmed in practice, there is little theoretical insight into this matter so far.

An important special case of the problems considered in this work are matrix equations of the form $\mathbf{A}(U) = B$ for a linear operator $\mathbf{A} : \mathbb{R}^{M\times N} \to \mathbb{R}^{M\times N}$. Clearly, any such operator can be written in the form

$$\mathbf{A}(U) = A_1^{(1)} U A_1^{(2)} + A_2^{(1)} U A_2^{(2)} + \cdots + A_{r_\mathbf{A}}^{(1)} U A_{r_\mathbf{A}}^{(2)}, \qquad A_i^{(1)} \in \mathbb{R}^{M\times M}, \quad A_i^{(2)} \in \mathbb{R}^{N\times N}$$

for some $r_\mathbf{A} \leq MN$. For $r_\mathbf{A} = 1$ and invertible matrices $A_1^{(1)}$, $A_1^{(2)}$ the rank of the solution U equals the rank of B. This property does not hold for $r_\mathbf{A} \geq 2$ and one then considers the question of low-rank approximability of U, that is, the decay of its singular values. Particular attention has been paid to the case of a Lyapunov matrix equation

$$AU + UA^T = B$$

for a matrix B of low rank, which plays an important role in control and model reduction, see, e.g., [6]. A number of works [1,4,9,10,12,25,26] have been devoted to studying low-rank approximability for this problem. In particular, it has be shown that the singular values of U decay exponentially when A is symmetric positive definite. All existing proof techniques implicitly rely on the fact that the two operators $U \mapsto AU$ and $U \mapsto UA^T$ commute. In particular, this allows for the simultaneous diagonalization of both operators, which greatly simplifies the approximation problem. When this commutativity property is lost, these techniques fail. For example, only partial results [5,21] are available so far for the innocently looking modification

$$AU + UA^T + CUC^T = B$$

for general matrix C, which plays a role in bilinear and stochastic control. This indicates that we cannot expect to obtain exponential singular value decay for such generalizations.

In general, we consider linear systems and eigenvalue problems of the form

$$\mathbf{Au} = \mathbf{b}, \qquad \mathbf{Au} = \lambda \mathbf{u}, \tag{1}$$

where $\mathbf{A}$ is a self-adjoint positive definite and bounded linear operator on a tensor product $H_1 \otimes \cdots \otimes H_d$ of Hilbert spaces H_μ, $\mu = 1, \dots, d$. We will study the low-rank approximability of the solution $\mathbf{u} \in H_1 \otimes \cdots \otimes H_d$ in certain tensor network formats, such as the tensor train format [22] (matrix product states [24]) and the hierarchical Tucker