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Functional identities in upper triangular matrix rings

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ABSTRACT

Let R be a subring of a ring Q, both having the same unity. We prove that if R is a d-free subset of Q, then the upper triangular matrix ring $T_n(R)$ is a d-free subset of $T_n(Q)$ for any $n \in \mathbb{N}$.

1. Introduction

The theory of functional identities (FI) was initiated by Matej Brešar at the beginning of the 1990s and it has been later developed mainly by Konstantin I. Beidar, M. Brešar, Mikhail A. Chebotar, and Wallace S. Martindale III. This theory has turned out to be a useful tool for solving different problems in many areas. A concise and comprehensive account of the theory and its applications can be found in the book [2]. Let R be a nonempty subset of an associative unital ring Q with center $Z(Q)$. Suppose that $m \in \mathbb{N}$

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and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \dots, m\}$. Let $E_i, F_j : R^{m-1} \to Q$ be arbitrary maps. We shall consider the following functional identities:

$$\sum_{i \in I} E_i(\bar{x}_m^i)x_i + \sum_{j \in J} x_j F_j(\bar{x}_m^j) = 0 \quad \text{for all } \bar{x}_m \in R^m \tag{1.1}$$

and

$$\sum_{i \in I} E_i(\bar{x}_m^i)x_i + \sum_{j \in J} x_j F_j(\bar{x}_m^j) \in Z(Q) \quad \text{for all } \bar{x}_m \in R^m, \tag{1.2}$$

where $\bar{x}_m = (x_1, \dots, x_m)$ and $\bar{x}_m^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$. Identities (1.1) and (1.2) are basic examples of FIs. The maps E_i, F_j are considered as unknowns and the central problem in the theory of functional identities is to find solutions of (1.1) (resp. (1.2)), that is, to describe the form of maps E_i, F_j satisfying (1.1) (resp. (1.2)). It turns out that there is always a solution of (1.1) (resp. (1.2)), no matter what R and Q are. Namely, let

$$p_{ij} : R^{m-2} \to Q, \quad i \in I, \ j \in J, \ i \neq j,$$
$$\lambda_k : R^{m-1} \to Z(Q), \quad k \in I \cup J,$$

be arbitrary maps. Suppose that

$$E_i(\bar{x}_m^i) = \sum_{\substack{j \in J \\ j \neq i}} x_j p_{ij}(\bar{x}_m^{ij}) + \lambda_i(\bar{x}_m^i), \quad i \in I,$$
$$F_j(\bar{x}_m^j) = -\sum_{\substack{i \in I \\ i \neq j}} p_{ij}(\bar{x}_m^{ij})x_i - \lambda_j(\bar{x}_m^j), \quad j \in J,$$
$$\lambda_k = 0 \ \text{ if } \ k \notin I \cap J, \tag{1.3}$$

where $\bar{x}_m^{ij} = \bar{x}_m^{ji} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots x_m)$. One can easily verify that (1.3) is indeed a solution of both (1.1) and (1.2). We say that (1.3) is a *standard solution* of (1.1) and (1.2). The case when one of the subsets I, J is empty is also included. Namely, according to the usual convention, that the sum over the empty set is zero, (1.1) can be rewritten as

$$\sum_{i \in I} E_i(\bar{x}_m^i)x_i = 0 \quad \text{for all } \bar{x}_m \in R^m \tag{1.4}$$

if $J = \emptyset$ or as

$$\sum_{j \in J} x_j F_j(\bar{x}_m^j) = 0 \quad \text{for all } \bar{x}_m \in R^m \tag{1.5}$$