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Yet another characterization of solutions of the Algebraic Riccati Equation

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ABSTRACT

This paper deals with a characterization of the solution set of algebraic Riccati equation (ARE) (over reals) for both controllable and uncontrollable systems. We characterize all solutions using simple linear algebraic arguments. It turns out that solutions of ARE of maximal rank have lower rank solutions encoded within it. We demonstrate how these lower rank solutions are encoded within the full rank solution and how one can retrieve the lower rank solutions from the maximal rank solution. We characterize situations where there are no full rank solutions to the ARE. We also characterize situations when the number of solutions to the ARE is finite, when they are infinite and when they are bounded. We also explore the poset structure on the solution set of ARE, which in some specific cases turns out to be a lattice which is isomorphic to lattice of invariant subspaces of a certain matrix. We provide several examples that bring out the essence of these results.

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1. Introduction and preliminaries

Algebraic Riccati equation occurs naturally in control theory, filtering, numerical analysis and many other engineering applications. In optimal control, algebraic Riccati equation (ARE) arises in infinite horizon continuous time LQR problem. ARE is also related to power method, QR factorization in matrix computations [3,10], spectral factorization [17,7,8]. Riccati equation shows up in Kalman filters too [13]. Refer [15] for Riccati equation arising in linear quadratic control problem. In [17], solutions of ARE are used in the study of acausal realizations of stationary processes. Further it is shown how AREs are involved in spectral factorization and in balancing algorithm (related to stochastic balancing) in [17]. In [7,8] solutions ARE are used for parametrization of minimal spectral factors. In [6], solutions of ARE are used to obtain parametrization of minimal stochastic realizations. For a treatment on discrete-time ARE, refer [9,24]. Recently study of ARE has appeared in papers on behavioral theory of systems [4,16]. For recent applications of Riccati equations to fluid queues models and transport equations refer [2] and the references therein. For more literature on Riccati equation, refer [1].

We concentrate on the ARE of the form $-A^TK - KA - Q + KBB^TK = 0$ where A, B, Q are real constant matrices having dimensions $n \times n$, $n \times m$ and $n \times n$ respectively with Q being symmetric. AREs with $Q = 0$ are known in the literature as homogeneous ARE. One of the methods for solving AREs involves using the eigenvectors of the Hamiltonian matrix H [13,3,21]. There are several other methods for solving the ARE (see [21,23]). Details about constructing solutions of ARE can be found in Chapters 2 and 3 of [3] and in Chapters 7–11 of [13]. It has been proved in [21] that if (A, B) is controllable, then there is a one-to-one correspondence between real and symmetric solutions K of ARE and n-dimensional H-invariant subspaces which are complementary to the span of $\begin{bmatrix} 0 \\ I \end{bmatrix}$ satisfying some special properties. It is known that if (A, B) pair is controllable and column span of $2n \times n$ matrix $\begin{bmatrix} U \\ V \end{bmatrix}$ is an H-invariant subspace satisfying some special property (i.e. the subspace being Lagrangian), then U is invertible and $K = VU^{-1}$ gives a solution of ARE [13,20].

From the literature, we know that a solution of ARE exists if there exists an n-dimensional Lagrangian H-invariant subspace. We assume that this is the case and fix an arbitrary solution K_0 of the ARE. Let $K = K_0 + X$ where X can be thought of as a perturbation from K_0. We can then re-write $-A^TK - KA - Q + KBB^TK$ as

$$\begin{aligned} &= -A^T(K_0 + X) - (K_0 + X)A - Q + (K_0 + X)BB^T(K_0 + X) \\ &= -A^TK_0 - K_0A - Q + K_0BB^TK_0 - A^TX - XA + K_0BB^TX \\ &\quad + XBB^TK_0 + XBB^TX \end{aligned}$$