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A new eigenvalue inclusion set for tensors and its applications

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ABSTRACT

A new tensor eigenvalue inclusion set is given, and proved to be tighter than those in L.Q. Qi (2005) [18] and C.Q. Li, Y.T. Li, X. Kong (2014) [12]. In addition, we study the eigenvalues lying on the boundary of the eigenvalue inclusion set provided by Qi (2005) for weakly irreducible tensors. As applications, we give new bounds for the spectral radius of nonnegative tensors, some sufficient conditions for a tensor to be a strong M-tensor and some inequalities to identify the positive definiteness for an even-order real supersymmetric tensor.

1. Introduction

We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order m dimension n, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$, respectively), if its entries

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$$a_{i_1 \cdots i_m} \in \mathbb{C}\ (\mathbb{R}),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, \ldots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. Furthermore, if $\mathcal{A}$ is real and x is a nonzero real vector, then λ is real and we call λ an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with the H-eigenvalue λ [18]. And the spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of $\mathcal{A}$, that is, the set containing all eigenvalues of $\mathcal{A}$. We say that $\mathcal{A}$ is nonnegative [11] if $\mathcal{A}$ is real and every of its entries $a_{i_1 \cdots, i_m} \geq 0$. The tensor $\mathcal{A}$ is called supersymmetric [7,18] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. Let $N = \{1, 2, \ldots, n\}$ and $\delta_{i_1 i_2 \cdots i_m}$ denote the Kronecker symbol for the case of m indices [18], that is, $\delta_{i_1 i_2 \cdots i_m} = 1$ if $i_1 = \cdots = i_m$, otherwise 0.

Eigenvalue problems of tensors have a wide range of practical applications, such as best-rank one approximation in data analysis [10,20,28], higher order Markov chains [15] and positive definiteness of even-order multivariate forms in automatical control [16], blind source separation [7], magnetic resonance imaging [22,23], molecular conformation [3], etc. Recently, tensor eigenvalues and eigenvectors have received much attention in the literature [1,8,9,11,15,18–22,25].

Due to the nonlinear nature of the eigenvalue problem for tensors, methods in matrix theory cannot be applied directly. However, many important results on the eigenvalue problem of matrices have been successfully extended to higher order tensors; see [1,2,13,15,18,19,21,26,27].