Linear Algebra and its Applications 481 (2015) 94–106

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Tropical spectral theory of tensors

Emmanuel Tsukerman

Department of Mathematics, University of California, Berkeley, CA 94720-3840, United States

ARTICLE INFO

Article history:
Received 10 December 2014
Accepted 22 April 2015
Available online 14 May 2015
Submitted by J.M. Landsberg

MSC:
14T05
15A69
15A18

Keywords:
Tropical geometry
Tensors
Eigenvalues

ABSTRACT

We introduce and study tropical eigenpairs of tensors, a generalization of the tropical spectral theory of matrices. We show the existence and uniqueness of an eigenvalue. We associate with a tensor a directed hypergraph and define a new type of cycle on such a hypergraph, which we call an H-cycle. The eigenvalue of a tensor turns out to be equal to the minimal normalized weighted length of H-cycles of the associated hypergraph. We show that the eigenvalue can be computed efficiently via a linear program.

1. Background

A tensor of order m and rank n is an array $A = (a_{i_1\cdots i_m})$ of elements of a field or semifield K (which we shall take to be $\mathbb{R}$ or $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$), where $1 \leq i_1, \ldots, i_m \leq n$. In ordinary arithmetic, given $x \in \mathbb{R}^n$, we define

$$(Ax^{m-1})_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

E-mail address: e.tsukerman@berkeley.edu.

http://dx.doi.org/10.1016/j.laa.2015.04.030

Here A is being contracted with $x^{\otimes(m-1)}$, the $(m-1)$st tensor power of x. An H-eigenpair [9] of a tensor is defined as follows. Define $x^{[m-1]} = (x_i^{m-1})_i$. Then an H-eigenpair is a pair $(x, \lambda) \in \mathbb{P}^{n-1} \times \mathbb{R}$ such that

$$Ax^{m-1} = \lambda x^{[m-1]}.$$

Let A be an $n \times n$ matrix with entries in the tropical semiring $(\mathbb{R}, \oplus, \odot)$. We shall take $\oplus$ to be min throughout. An eigenvalue of A is a number λ such that

$$A \odot \boldsymbol{v} = \lambda \odot \boldsymbol{v}.$$

The nature of tropical eigenpairs is understood in the setting of matrices [10,12] but a survey of the literature shows no prior research on tropical eigenpairs of tensors.

Definition 1.1. A *tropical H-eigenpair* for a tensor $(a_{i_1 \cdots i_m}) \in \mathbb{R}^{n^m}$ of order m and rank n is a pair $(x, \lambda) \in \mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1) \times \mathbb{R}$ such that

$$\bigoplus_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} \odot x_{i_2} \odot \cdots \odot x_{i_m} = \lambda \odot x_i^{\odot(m-1)}, \quad i = 1, 2, \ldots, n. \tag{1}$$

We call x a *tropical H-eigenvector* and λ a *tropical H-eigenvalue*.

In the classical setting, several other definitions of eigenpairs of tensors exist. For instance, an E-eigenpair is defined via the condition

$$Ax^{m-1} = \lambda x.$$

We define the tropicalization here in an analogous manner. In this paper, we focus on H-eigenpairs and only discuss E-eigenpairs for purposes of comparison. Note that the notions of E- and H-eigenpairs coincide with the usual notion of eigenpairs of matrices in the case of matrices.

Example 1.2. Take $n = 2$ and $m = 3$. Then a tropical H-eigenpair (x, λ) satisfies

$$\min\{a_{111} + 2x_1, a_{112} + x_1 + x_2, a_{121} + x_2 + x_1, a_{122} + 2x_2\} = \lambda + 2x_1$$
$$\min\{a_{211} + 2x_1, a_{212} + x_1 + x_2, a_{221} + x_2 + x_1, a_{222} + 2x_2\} = \lambda + 2x_2.$$

Define a partial symmetrization map PSym on tensors via

$$(\mathrm{PSym}\, A)_{i_1, \ldots, i_n} := \min_{\sigma \in S\{2, \ldots, m\}} A_{i_1, i_{\sigma(2)}, \ldots, i_{\sigma(n)}},$$

where $S\{2, \ldots, m\}$ is the set of permutations of $\{2, \ldots, m\}$. Let $a'_{i_1, \ldots, i_n} := (\mathrm{PSym}\, A)_{i_1, \ldots, i_n}$. Observe that