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# Structure theorems for star-commuting power partial isometries

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ABSTRACT

We give a new formulation and proof of a theorem of Halmos and Wallen on the structure of power partial isometries on Hilbert space. We then use this theorem to give a structure theorem for a finite set of partial isometries which star-commute: each operator commutes with the others and with their adjoints.



## 1. Introduction

The Wold–von Neumann theorem says that every isometry on a Hilbert space is a direct sum of a unitary operator and unilateral shifts. Halmos and Wallen [5] proved a similar result for *power partial isometries*: operators such that all positive powers are

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partial isometries. Their theorem says that every power partial isometry is a direct sum of a unitary operator, some unilateral (forward) shifts, some backward shifts and some truncated shifts on finite-dimensional spaces.

There has been recurring interest in analogues of the Wold–von Neumann theorem for families of commuting isometries [13,12,1,8]. It has been known for many years that the most satisfactory results are those for families which star-commute, in the sense that each isometry commutes with the other isometries and with their adjoints (see [12,3, 11], and the extensive references in [11]). There have been similar results for pairs of star-commuting power partial isometries based on the Halmos–Wallen theorem [4,2].

Here we give a modern formulation of the Halmos–Wallen theorem in terms of tensor products, and use it to prove a structure theorem for finite families of star-commuting power partial isometries. This last result seems to be new, perhaps even for operators on a finite-dimensional space, and for isometries it looks quite different from the existing versions. For pairs of power partial isometries, it also looks quite different from the decomposition in [2], and the tensor-product decompositions obtained in [4, §3], which are for special cases where the individual Halmos–Wallen decompositions have a single summand, follow from our result.

## 2. The Halmos–Wallen theorem

We use the basic properties of partial isometries, as discussed in [9, §A.1], for example. We also need to know that if $V$ and $W$ are partial isometries, then $VW$ is a partial isometry if and only the initial projection $V^*V$ commutes with the range projection $WW^*$ [5, Lemma 2]. An operator $T$ is a *power partial isometry* if $T^n$ is a partial isometry for all $n \geq 0$, and then $\{T^nT^{*n}\} \cup \{T^{*n}T^n\}$ is a commuting family of projections. (We have just established a notational convention: $T^{*n}$ means $(T^*)^n$, and we allow also $T^{*n-m}$ for $(T^*)^{n-m}$ with $n \geq m$.)

Examples of power partial isometries include unitary operators, the unilateral shift $S$ on $\ell^2$, the backward shift $S^*$, and the truncated shifts $J_p$ defined in terms of the usual basis for $\mathbb{C}^p$ by $J_pe_n = e_{n+1}$ for $n < p$ and $J_pe_p = 0$. (Notice that $p \geq 1$, and we include $J_1 = 0$.) The Halmos–Wallen theorem says that every power partial isometry can be constructed from these examples.

Our models involve tensor products of Hilbert spaces and bounded operators on them. All we need for the present theorem are the relatively elementary properties covered in [7, §2.6] and [10, §2.4 and §B.1], for example. (Though in the next section we use some less elementary facts about tensor products of $C^*$-algebras.)

**Theorem 2.1** *(Halmos and Wallen). Let $T$ be a power partial isometry on a Hilbert space $H$, and let $P$ and $Q$ be the orthogonal projections on $\bigcap_{n=1}^{\infty} T^nH$ and $\bigcap_{n=1}^{\infty} T^{*n}H$ respectively. Then $PQ = QP$ and the subspaces $H_u := PQH$, $H_s := (1 - P)QH$, $H_b := (1 - Q)PH$ and*