Linear Algebra and its Applications 481 (2015) 280–287

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A contribution to the Aleksandrov conservative distance problem in two dimensions

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ARTICLE INFO

Article history:
Received 28 January 2015
Accepted 4 May 2015
Available online 15 May 2015
Submitted by P. Semrl

MSC:
primary 46B04, 46B20, 52A10
secondary 51M05, 52C25

Keywords:
Aleksandrov conservative distance problem
Beckman–Quarles type space
Isometry
Unit distance preserving mapping

ABSTRACT

Let E be a two-dimensional real normed space. In this paper we show that if the unit circle of E does not contain any line segment such that the distance between its endpoints is greater than 1, then every transformation $\phi: E \to E$ which preserves the unit distance is automatically an affine isometry. In particular, this condition is satisfied when the norm is strictly convex.

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[1] The author was supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences.

http://dx.doi.org/10.1016/j.laa.2015.05.005
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1. Introduction

In 1953 F.S. Beckman and D.A. Quarles characterized isometries of n-dimensional Euclidean spaces under a surprisingly mild condition when $n \geq 2$ (see [2] or [3,4,9] for alternative proofs). Namely, they managed to show that every transformation $\phi: \mathbb{R}^n \to \mathbb{R}^n$ which preserves unit Euclidean distance in one direction is an (affine) isometry. They also noted that on $\mathbb{R}$ or on an infinite dimensional, real Hilbert space the same conclusion fails.

Many mathematicians have been trying to generalize this beautiful theorem. The problem of characterizing those finite dimensional real normed spaces E such that every transformation $\phi: E \to E$ which preserves the unit distance in one direction is an isometry was raised, in this general form, by A.D. Aleksandrov and hence it is called the Aleksandrov conservative distance problem (see [1]). In the literature these spaces are also called Beckman–Quarles type spaces. As far as we know, the original version of Aleksandrov problem was solved only for a few concrete normed spaces (see [22] concerning p-norms, and [13] where the norm is not strictly convex), all of them are two-dimensional. Some general results are known for modified versions, for instance in [5] W. Benz and H. Berens investigated the case when the transformation preserves distance 1 and n for some $n \in \mathbb{N}$, $n > 1$. We also mention the paper [18] of T.M. Rassias and P. Šemrl where they assumed that ϕ is onto and it preserves distance 1 in both directions. They showed that in this case ϕ is not very far from being an isometry. Several other results are known which are connected to the Aleksandrov problem. The reader can find a number of them in the References, see e.g. [4,6,8,10–12,15,19,20].

The original version remained unsolved even for the very special case when $\dim E = 2$ and the norm is strictly convex. Here we present a unified approach which solves the Aleksandrov problem in two dimensions for a much larger class of norms, which we will call URTC-norms. Let us point out that the naive conjecture that every at least two but finite dimensional normed space is a Beckman–Quarles type space, is false. However, as far as we know, counterexamples are only known in the simple case when the unit ball of the norm is a linear image of a cube (see [17]).

2. Auxiliary definitions and statement of the main result

Since we will consider only two-dimensional normed spaces over $\mathbb{R}$, we can investigate $\mathbb{R}^2$ endowed with a norm $\|\cdot\|$. We say that the norm is strictly convex, if its sphere S does not contain any non-degenerated line segment. If three points $a, b, c \in \mathbb{R}^2$ satisfy $d = \|a - b\| = \|b - c\| = \|c - a\|$ for some $d > 0$, then these points are said to be in a regular d-position. We introduce the following notion.

Definition 1. We call $\|\cdot\|$ a URTC-norm (unique regular triangle constructibility) if for every $a, b \in \mathbb{R}^2$, $\|a - b\| = 1$ the equation system

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