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Secants of minuscule and cominuscule minimal orbits

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ABSTRACT

We study the geometry of the secant and tangential variety of a cominuscule and minuscule variety, e.g. a Grassmannian or a spinor variety. Using methods inspired by statistics we provide an explicit local isomorphism with a product of an affine space with a variety which is the Zariski closure of the image of a map defined by generalized determinants. In particular, equations of the secant or tangential variety correspond to relations among generalized determinants. We also provide a representation theoretic decomposition of cubics in the ideal of the secant variety of any Grassmannian.

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1. Introduction

The aim of the article is to investigate the properties of the secant variety of the minimal orbit in a minuscule and cominuscule representation of a semisimple complex Lie group. The prototypical examples of such varieties are the Grassmannians. The Grassmannian of k dimensional subspaces of an n dimensional vector space V is the image of the map

$$\{\text{nondegenerate } k \times n \text{ matrices}\} \to \mathbb{P}(\bigwedge^k V)$$
$$M \mapsto [\text{all maximal minors of } M].$$

Moreover, we can parameterize an affine open chart of the Grassmannian by

$$\{k \times (n-k) \text{ matrices}\} \to \mathbb{A}^{\binom{n}{k}-1} \subset \mathbb{P}(\bigwedge^k V)$$
$$M \mapsto (\text{all minors of } M).$$

In particular, one can consider the Plücker relations that define the Grassmannian, as quadratic relations among minors, coming from the Laplace expansion of the determinant.

We generalize these classical observations to the tangential and secant variety, by providing analogous local parameterizations. Recall that the tangential variety is the union of all tangent lines to the variety, while the secant variety is the Zariski closure of the union of the bisecant lines. It turns out that the tangential variety is locally isomorphic to a product of an affine space by the Zariski closure of the variety M parameterized by all minors of degree at least two of a generic matrix. The secant variety is locally isomorphic to a product of an affine space by the cone over M. In particular, the equations of the tangential (resp. secant) variety correspond to (resp. homogeneous) relations among minors of degree at least 2.

Our method is inspired by the "cumulant trick" coming from statistics. Given probability distributions, the statisticians compute general moments and cumulants. The formulas for those were the inspiration to define two triangular automorphisms of the affine space. This method has had other successful applications [26,23]. Recently further interesting results were obtained in [3].

Furthermore, using generalized determinants, we are able to extend our results to all varieties that are both minuscule and cominuscule obtaining our main theorem. In order to prove our results we present formulas for the generalized determinants for a sum and a generalized Laplace expansion in Lemmas 2.5 and 2.6. The following setting also includes the spinor varieties and the two exceptional Hermitian symmetric spaces. The equations of the secant and tangential variety correspond to the relations among the Pfaffians.