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A uniqueness result for linear complementarity problems over the Jordan spin algebra

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ABSTRACT

Given a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$ with the (corresponding) symmetric cone K, a linear transformation $L : V \to V$ and $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, q)$ is to find a vector $x \in V$ such that

$$x \in K, \quad y := L(x) + q \in K \quad \text{and} \quad x \circ y = 0.$$

To investigate the global uniqueness of solutions in the setting of Euclidean Jordan algebras, the P-property and its variants of a linear transformation were introduced in Gowda et al. (2004) [3] and it is shown that if $\mathrm{LCP}(L, q)$ has a unique solution for all $q \in V$, then L has the P-property but the converse is not true in general. In the present paper, when $(V, \circ, \langle\cdot,\cdot\rangle)$ is the Jordan spin algebra, we show that $\mathrm{LCP}(L, q)$ has a unique solution for all $q \in V$ if and only if L has the P-property and L is positive semidefinite on the boundary of $\mathcal{K}$.

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1. Introduction

For $n > 1$, consider $\mathbb{R}^n$ with the usual inner-product $\langle x, y\rangle := x^T y$. We write a vector x in $\mathbb{R}^n$ by $[x_0, \bar{x}]^T$, where $x_0 \in \mathbb{R}$ and define the Jordan product of any two n-vectors x and y by

$$x \circ y := \begin{bmatrix} x^T y \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}.$$

The triple $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle)$ is called *Jordan-spin algebra*. The cone of squares

$$\mathcal{K}^n := \{x \in \mathbb{R}^n : x_0 \geq 0, \quad \bar{x}^T \bar{x} \leq x_0^2\} = \{x \circ x : x \ \in \mathbb{R}^n\}$$

is the well-known second-order cone or the Lorentz cone. The most interesting case happens when $\mathcal{K}^n$ is non-polyhedral which is true iff $n > 2$. Henceforth, we fix $n > 2$ and simply use $\mathcal{K}$ to denote $\mathcal{K}^n$. Given an $n \times n$ real matrix M and a vector q in $\mathbb{R}^n$, the second-order cone linear complementarity problem SOLCP(M, q) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \ \text{ and } \ x \circ y = 0 (\Leftrightarrow x^T y = 0).$$

Complementarity problems appear in various areas that include game theory, optimization and economics. SOLCP is a classical example of a linear complementarity problem defined on a non-polyhedral cone. The Jordan spin algebra associated with the second-order cone has rank two and has extra properties which allow us to go beyond the general study of complementarity problems. The text of Faraut and Koranyi [2] covers the foundations of Euclidean Jordan algebra.

Definition 1. We will say that an $n \times n$ real matrix M has the *Globally Uniquely Solvable* (GUS) property, if SOLCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$.

In general it is very difficult to verify whether a linear transformation has the GUS-property. Investigations on GUS-property of a linear transformation in Euclidean Jordan algebras are found in Gowda and Sznajder [4]. One of the fundamental problems in SOLCP is to find conditions that characterize the GUS-property of an $n \times n$ matrix. The problem can be posed in a more general setting.

Given a finite dimensional real Hilbert space $\mathcal{H}$, and a closed convex cone $\mathcal{C}$ in $\mathcal{H}$, a linear transformation $T : \mathcal{H} \to \mathcal{H}$, and a vector $q \in \mathcal{H}$, the (cone) linear complementarity problem LCP$(T, \mathcal{H}, \mathcal{C})$ is to find a vector $x \in \mathcal{H}$ such that

$$x \in \mathcal{C}, \quad y := Tx + q \in \mathcal{C}^* \ \text{ and } \ \langle x, y\rangle = 0,$$