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Real numerical shadow and generalized B-splines

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A B S T R A C T

Restricted numerical shadow $P_A^X(z)$ of an operator A of order N is a probability distribution supported on the numerical range $W_X(A)$ restricted to a certain subset X of the set of all pure states – normalized, one-dimensional vectors in $\mathbb{C}^N$. Its value at point $z \in \mathbb{C}$ equals the probability that the inner product $\langle u|A|u\rangle$ is equal to z, where u stands for a random complex vector from the set X distributed according to the natural measure on this set, induced by the unitarily invariant Fubini–Study measure. For a Hermitian operator A of order N we derive an explicit formula for its shadow restricted to real states, $P_A^{\mathbb{R}}(x)$, show relation of this density to the Dirichlet distribution and demonstrate that it forms a generalization of the B-spline. Furthermore, for operators acting on a space with tensor product structure, $\mathcal{H}_A \otimes \mathcal{H}_B$, we analyze the shadow restricted to the set of maximally entangled states and derive distributions for operators of order $N = 4$.

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1. Introduction

Consider a complex square matrix A or order N. Its standard *numerical range* is defined as the following subset of the complex plane,

$$W(A) = \{\langle u|A|u\rangle : u \in \mathbb{C}^N, \|u\| = 1\},$$

where u denotes a normalized complex vector in $\mathcal{H}_N$. Due to the Toeplitz–Hausdorff theorem this set is convex, while for a Hermitian A it forms an interval belonging to the real axis – see e.g. [1–3].

Among numerous generalizations of this notion we will be concerned with the *restricted numerical range*,

$$W_X(A) = \{\langle u|A|u\rangle : u \in \omega_X\}, \tag{1}$$

where ω_X forms a certain subset of the set ω of normalized complex vectors of size N. For instance, one can choose ω_X as the set of all real vectors, and analyze the 'real shadow' of A, denoted by $W_{\mathbb{R}}(A)$. For an operator A acting on a composed space, one studies also numerical range restricted to tensor product states, $W_{\otimes}(A)$, and the range $W_E(A)$ restricted to maximally entangled states [4,5]. It is worth to emphasize a crucial difference with respect to the standard notion: the restricted numerical range needs not to be convex.

In order to define a probability measure supported on numerical range of $W(A)$ it is sufficient to consider the uniform measure on the sphere S^{2N-1} and the measure induced by the map $u \to \langle u|A|u\rangle \in W(A)$ [6,7]. Alternatively, one considers the space of quantum states – equivalence classes of normalized vectors in $\mathbb{C}^N$, which differ by a complex phase, $u \sim e^{i\alpha}u$, and works with the Haar measure invariant under the action of the unitary group [8]. For any matrix A one defines in this way a probability measure $P_A(z)$ supported on $W(A)$ and called *numerical shadow* [6] or *numerical measure* [7]. The former name is inspired by the fact that for a normal matrix this measure can be interpreted as a shadow of an uniformly covered $(N-1)$ dimensional regular simplex projected on a plane [8,9] (see Fig. 1). In a similar fashion, one can consider numerical shadow of matrices over the quaternion field, defined as the pushforward measure of the uniform measure on the sphere S^{4N-1}.

Even though several papers on numerical shadow were published during the last five years [6–8], the idea to associate with the numerical range a probability measure is much older: as described in a recent review by Holbrook [10] it goes back to the early papers of Davis [1].

Another variant of the numerical shadow of A can be obtained by taking random points from the subset ω_X of the set of pure states. The corresponding probability measure $P_A^X(z)$, called *restricted numerical shadow* [11] (see Fig. 1 for an example of restriction to real pure states), is by definition supported in restricted numerical range

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