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The effect of finite rank perturbations on Jordan chains of linear operators

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A general result on the structure and dimension of the root subspaces of a linear operator under finite rank perturbations is proved: The increase of dimension from the kernel of the n-th power to the kernel of the $(n + 1)$-th power of the perturbed operator differs from the increase of dimension of the kernels of the corresponding powers of the unperturbed operator by at most the rank of the perturbation. This bound is sharp.

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1. Introduction

Perturbation theory for linear operators and their spectra is one of the main objectives in operator theory and functional analysis, with numerous applications in mathematics, physics and engineering sciences. In many approaches compact perturbations and perturbations small in size are investigated, e.g. when stability properties of the index, nullity and deficiency of Fredholm and semi-Fredholm operators are analyzed. A widely used and well-known fact on the effect of compact perturbations is the following: If S and T are bounded operators in a Banach space, $K = T - S$ is compact and $\lambda \in \mathbb{C}$ is such that $S - \lambda$ is Fredholm, then also $T - \lambda$ is Fredholm and the Fredholm index is preserved. In particular, since $\ker(S - \lambda)$ and $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$ are finite dimensional the same is true for $\ker(T - \lambda)$ and $\ker(T - \lambda)^{n+1}/\ker(T - \lambda)^n$. However, for such an arbitrary compact perturbation K there exists no bound on the dimensions of $\ker(T - \lambda)$ or $\ker(T - \lambda)^{n+1}/\ker(T - \lambda)^n$ in terms of $\ker(S - \lambda)$ or $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$, respectively. The situation is different when the perturbation is not only compact but of finite rank.

In the present note we consider general linear operators S and T in a vector space X such that T is a finite rank perturbation of S. It follows easily that the dimensions of $\ker(S - \lambda)$ and $\ker(T - \lambda)$ differ at most by k if the perturbation $K = S - T$ is an operator with $\operatorname{rank}(K) = k$. Our main objective is to explore the connections between the kernels of consecutive higher powers of $S - \lambda$ and $T - \lambda$ in more detail, and to prove the following general result on the structure and dimensions of the root subspaces under finite rank perturbations: Given a linear operator S in X, consider the space $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$. Its dimension coincides with the number of linearly independent Jordan chains of S at λ of length at least $n + 1$. It then turns out that the change of the number of these Jordan chains of S at λ under a rank k perturbation is bounded by k,

$$\left|\dim\left(\frac{\ker(S - \lambda)^{n+1}}{\ker(S - \lambda)^n}\right) - \dim\left(\frac{\ker(T - \lambda)^{n+1}}{\ker(T - \lambda)^n}\right)\right| \le k, \tag{1.1}$$

and this bound is sharp, see Theorem 2.2 and Example 2.3. Here S and T are defined on subspaces of X and the finite rank perturbation is interpreted in a generalized sense, see Hypothesis 2.1. In particular, our assumptions allow to treat unbounded operators in Banach spaces and finite rank perturbations in resolvent sense. We also emphasize that the dimensions of the root subspaces of the operators S and T may be infinite, and that a finite rank perturbation may turn points from the resolvent set of S into eigenvalues of infinite algebraic multiplicity of T; cf. Example 2.5.

If X is finite dimensional, then S and T are matrices and (1.1) was already proved by S.V. Savchenko in [10, Lemma 2], see also [1,2,5–9] for related results on so-called generic perturbations of matrices. Moreover, there exists a lower bound for the dimension of the root subspace of the perturbed operator T in terms of the dimension of the root subspace of S and the length of the Jordan chains of S at λ; cf. [3,10]. Such a result was also proved