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Group-theoretic constructions of erasure-robust frames

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A B S T R A C T

Many emerging frame theories and compressed sensing problems involve estimating the singular values of a combinatorially large number of submatrices. Such problems include explicitly constructing matrices with the restricted isometry property (RIP) and numerically erasure robust frames (NERFs), both of which seemingly requiring an enormous amount of computation in even low-dimensional examples. In this paper, we focus on NERFs which are the latest invention in a long line of research concerning the design of linear encoders that are robust against data loss. We begin by examining a subtle difference between the definition of a NERF and that of an RIP matrix, one that allows us to introduce a new computational trick for quickly estimating NERF bounds. In short, we estimate these bounds by evaluating the frame analysis operator at every point of an ε-net for the unit sphere. We then borrow ideas from the theory of group frames to construct explicit frames and ε-nets with such high degrees of symmetry that the requisite number of operator evaluations is greatly reduced. We conclude with

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numerical results, using these new ideas to quickly produce reasonable estimates of NERF bounds which would otherwise not be possible with existing methods.
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1. Introduction

Throughout this work, let Φ denote a short, real $M \times N$ matrix; though some of the results presented here generalize to the complex setting, others do not. That is, let $M \leq N$ and let $\Phi = [\varphi_1 \ \cdots \ \varphi_N]$ where $\{\varphi_n\}_{n=1}^N \subseteq \mathbb{R}^M$ are the columns of Φ. For any $\mathcal{K} \subseteq \{1, \ldots, N\}$, define $K := |\mathcal{K}|$ and consider the $M \times K$ submatrix $\Phi_\mathcal{K}$ of Φ obtained by including only the columns $\{\varphi_n\}_{n \in \mathcal{K}}$. Fixing K, both the restricted isometry property and numerically erasure-robust frames are defined in terms of the extreme singular values of $\Phi_\mathcal{K}$ as $\mathcal{K}$ ranges over all K-element subsets of $\{1, \ldots, N\}$; the key difference between the two depends on whether the $\Phi_\mathcal{K}$'s are tall ($K \leq M$) or short ($M \leq K \leq N$).

The purpose of this paper is to provide new deterministic constructions of *numerically erasure-robust frames* (NERFs): for $M \leq K \leq N$ and $0 < \alpha \leq \beta < \infty$, we say that $\{\varphi_n\}_{n=1}^N$ is a (K, α, β)-NERF for $\mathbb{R}^M$ if for any K-element subset $\mathcal{K} \subseteq \{1, \ldots, N\}$, we have

$$\alpha \|x\|^2 \leq \sum_{n \in \mathcal{K}} |\langle x, \varphi_n \rangle|^2 \leq \beta \|x\|^2, \quad \forall x \in \mathbb{R}^M. \tag{1}$$

This means that $\{\varphi_n\}_{n \in \mathcal{K}}$ is a frame for $\mathbb{R}^M$ with frame bounds α, β regardless of the choice of $\mathcal{K}$. Note (1) is equivalent to having $\alpha \leq \|\Phi_\mathcal{K}^* x\|^2 \leq \beta$ for all unit norm $x \in \mathbb{R}^M$, which in turn is equivalent to having all the eigenvalues of the subframe operator $\Phi_\mathcal{K} \Phi_\mathcal{K}^*$ lie in the interval $[\alpha, \beta]$.

Overall, when $\alpha \approx \beta$, we see that a NERF is a short matrix for which every *short* submatrix of a given size is well-conditioned. NERFs are the latest invention [11] in a long line of research [12,7,13,15,2] concerning the design of linear encoders that are robust against data loss. Here, one encodes $x \in \mathbb{R}^M$ as a higher-dimensional vector $\Phi^* x \in \mathbb{R}^N$ which is then transmitted in a channel with erasures and additive noise, yielding $y = \Phi_\mathcal{K}^* x + \varepsilon$, where $\mathcal{K}$ corresponds to the entries of y that were not erased. The problem is then to reconstruct x from y. In the standard least squares approach, this reconstruction is achieved by solving the normal equations $\Phi_\mathcal{K} \Phi_\mathcal{K}^* x = \Phi_\mathcal{K} y$. The numerical stability of this method depends heavily on the condition number of $\Phi_\mathcal{K} \Phi_\mathcal{K}^*$ which, in accordance with (1), has an upper bound of $\frac{\beta}{\alpha}$. As such, when designing NERFs, our goal is to make $K \geq M$ as small as possible—meaning we are robust against more erasures—while keeping $\frac{\beta}{\alpha}$ from becoming too large; this contrasts with the restricted isometry property (RIP), where the goal is to make $K \leq M$ as large as possible while satisfying