Linear Algebra and its Applications 479 (2015) 202–215

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Distributional chaos for the Forward and Backward Control traffic model ☆,☆☆

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ARTICLE INFO

Article history:
Received 4 December 2014
Accepted 8 April 2015
Available online 2 May 2015
Submitted by R. Brualdi

MSC:
47A16

Keywords:
Distributional chaos
C_0-semigroup
Devaney-chaos

ABSTRACT

The interest in car-following models has increased in the last years due to its connection with vehicle-to-vehicle communications and the development of driverless cars. Some non-linear models such as the Gazes–Herman–Rothery model were already known to be chaotic. We consider the linear Forward and Backward Control traffic model for an infinite number of cars on a track. We show the existence of solutions with a chaotic behaviour by using some results of linear dynamics of C_0-semigroups. In contrast, we also analyse which initial configurations lead to stable solutions.

☆ This paper is dedicated to Prof. Richard M. Aron on his 70th birthday for a whole life dedicated to Mathematics and for his contribution to make the Department of Mathematical Sciences of Kent State University become a mathematical hub for mathematicians all over the world.

☆☆ The first three authors were supported by MTM2013-47093-P. The first author was supported by the ERC grant HEVO no. 2177691. The third author was also supported by MTM2013-47093-P and by a grant from the FPU program of MEC (AP 2010-4361).

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http://dx.doi.org/10.1016/j.laa.2015.04.010
0024-3795/

Birth-and-death models
Car-following models

1. Introduction

In this paper we study chaotic traffic patterns described by car-following models. Several notions of chaos can be considered, such as the one of Devaney [19] or the one of distributional chaos by Schweizer and Smítal [41]. Devaney chaos consists of 3 ingredients: transitivity, existence of a dense set of periodic points, and sensitive dependence on the initial conditions. Many ways have been used in order to explain this last notion. Here we refer to the one by Ethan Hunt due its connection with traffic. *For instance, when you hit the brakes for a second, just tap them on the freeway, you can literally track the ripple effect of that action across a 200-mile stretch of road, because traffic has a memory*, see [1]. In other words, when considering a number of cars on a track, the behaviour of one of them can be transmitted and propagated to the ones in front (and behind) of it. The mathematical models used to described these interactions are known as *car-following* models.

The first ones were due to Greenshields [28,29] in the 1930s. Car-following models were perfectioned in the 50s and 60s by taking into account considerations involved in driving a motor vehicle on a lane [17], such as the difference between the velocities of a car and the car in front of it, a distance of a car with respect to the preceding one, or the driver's reaction time, see for instance [26,39]. An interested reader can find a historical evolution of these models in [14]. Recently, these models have attracted the interest of researchers thanks to the development of vehicle-to-vehicle (V2V) and of vehicle-to-infrastructure (V2I) communications, see e.g. [31]. These models contribute not only to the study of the possibility of allowing vehicles to "talk" or communicate with each other, but also to increase the efficiency of vehicles communication with the networks.

One of the simplest models is the *Quick-Thinking Driver* (QTD) model, which states that the acceleration of a car depends on its distance with respect to the car in front of it. With just two cars, with one of them following the other, one can even find chaos relating its dynamics with certain solutions of the logistic equation, see [35]. Nevertheless, it has already been known for decades that chaotic behaviours exist in traffic flow systems. Gazis, Herman, and Rothery developed for General Motors a generalized car-following model, known as the (GHR) model. The discontinuous behaviour of some of their solutions and the nonlinearity presented there suggested the existence of chaotic solutions for a certain range of input parameters, [27,40]. Later on, Disbro and Frame [20] showed the presence of chaos for (GHR) model without taking into account signals, bottlenecks, intersections, etc. or with a coordinated signal network. Chaos was also observed for a platoon of vehicles described by the traditional (GHR) model modified by adding a nonlinear inter-car separation dependent term, [2,3].

However, when taking an infinite number of cars on a lane, each of them following one another, even a linear simplification of these models can show chaotic phenomena. In [15]