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Characterization of k-commuting additive maps on rings ☆

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ABSTRACT

Let k be a positive integer and $\mathcal{R}$ a ring having unit 1. Denote by $\mathcal{Z}(\mathcal{R})$ the center of $\mathcal{R}$. Assume that the characteristic of $\mathcal{R}$ is not 2 and there is an idempotent element $e \in \mathcal{R}$ such that $\mathcal{R}$ satisfies $a\mathcal{R}e = \{0\} \Rightarrow a = 0$, $a\mathcal{R}(1-e) = \{0\} \Rightarrow a = 0$, $\mathcal{Z}(e\mathcal{R}e)_k = \mathcal{Z}(e\mathcal{R}e)$ and $\mathcal{Z}((1-e)\mathcal{R}(1-e))_k = \mathcal{Z}((1-e)\mathcal{R}(1-e))$. Then every additive map $f : \mathcal{R} \to \mathcal{R}$ is k-commuting if and only if $f(x) = \alpha x + h(x)$ for all $x \in \mathcal{R}$, where $\alpha \in \mathcal{Z}(\mathcal{R})$ and h is an additive map from $\mathcal{R}$ into $\mathcal{Z}(\mathcal{R})$. As applications, all k-commuting additive maps on prime rings and von Neumann algebras are characterized.

1. Introduction

Let $\mathcal{R}$ be an associative ring. For any elements $a, b \in \mathcal{R}$, we set $[a,b]_0 = a$, $[a,b]_1 = ab - ba$, and inductively $[a,b]_k = [[a,b]_{k-1}, b]$, where k is a positive integer. Denote by $\mathcal{Z}(\mathcal{R})$ the center of $\mathcal{R}$ and $\mathcal{Z}(\mathcal{R})_k = \{z \in \mathcal{R}\colon [z,a]_k = 0 \text{ for all } a \in \mathcal{R}\}$. Clearly, $\mathcal{Z}(\mathcal{R})_1 = \mathcal{Z}(\mathcal{R})$ and $\mathcal{Z}(\mathcal{R})_k \subseteq \mathcal{Z}(\mathcal{R})_m$ whenever $k \leqslant m$. Recall that an additive map

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$f : \mathcal{R} \to \mathcal{R}$ is k-commuting if $[f(a), a]_k = 0$ for all $a \in \mathcal{R}$; particularly, is commuting if $k = 1$, that is, $[f(a), a] = 0$ for all $a \in \mathcal{R}$.

The study of commuting maps and the relation with the commutativity of rings goes back to 1955 when Divinsky [9] proved that a simple Artinian ring is commutative if it has a commuting automorphism different from the identity map. This raises an interesting question of characterizing commuting additive maps on rings. Brešar in [4] proved that, if F is a commuting additive map from a von Neumann algebra $\mathcal{M}$ into itself, then there exist $Z \in \mathcal{Z}(\mathcal{M})$ and an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $F(A) = ZA + h(A)$ for all $A \in \mathcal{M}$. Later, Brešar [6] gave the same characterization of commuting additive maps on prime rings. For other results on various rings and algebras, see Refs. [5,8,14] and the references therein. Also, see a survey paper of Brešar [7] for a full account on commuting additive maps.

It is clear that the concept of k-commuting maps is a generalization of commuting maps. Generally speaking, every commuting map is k-commuting for any $k \geqslant 1$; but the converse is not true. Vukman [17] showed that, if there exists a nonzero derivation d on a prime ring $\mathcal{R}$ with characteristic not 2 such that $[d(x), x]_2 = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Also, by applying the theory of functional identities in rings, Beidar [2] and Martindale [3] studied the k-commuting additive maps respectively on prime rings and prime rings with involution. For other results on k-commuting additive maps, see [12,14,16] and the references therein.

Recently, the problem of characterizing k-commuting additive maps was studied by Du and Wang in [10], where they gave a concrete form of k-commuting additive maps on certain triangular algebras. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $f : \mathcal{U} \to \mathcal{U}$ a k-commuting additive map. If $\mathcal{U}$ satisfies the conditions: (i) $\mathcal{Z}(\mathcal{A})_k = \pi_A(\mathcal{Z}(\mathcal{U}))$, the natural projection from $\mathcal{U}$ onto $\mathcal{A}$, (ii) $\mathcal{Z}(\mathcal{B})_k = \pi_B(\mathcal{Z}(\mathcal{U}))$ and (iii) there exists $m_0 \in \mathcal{M}$ such that $\mathcal{Z}(\mathcal{U}) = \{a + b\colon\ a \in \mathcal{Z}(\mathcal{A}),\ b \in \mathcal{Z}(\mathcal{B}),\ am_0 = m_0 b\}$, then there exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$.

The purpose of the present paper is to consider the problem of characterizing k-commuting additive maps on general rings.

Let $\mathcal{R}$ be a ring having unit 1 and an idempotent element e, and let $\mathcal{Z}(\mathcal{R})$ denote the center of $\mathcal{R}$. Assume that the characteristic of $\mathcal{R}$ is not 2 and satisfies that $a\mathcal{R}e = \{0\} \Rightarrow a = 0$, $a\mathcal{R}(1-e) = \{0\} \Rightarrow a = 0$, $\mathcal{Z}(e\mathcal{R}e)_k = \mathcal{Z}(e\mathcal{R}e)$ and $\mathcal{Z}((1-e)\mathcal{R}(1-e))_k = \mathcal{Z}((1-e)\mathcal{R}(1-e))$. Let $f : \mathcal{R} \to \mathcal{R}$ be an additive map. We show that the following three statements are equivalent: (1) f is k-commuting for positive integer $k \geqslant 1$; (2) f is commuting; (3) there exist $\alpha \in \mathcal{Z}(\mathcal{R})$ and an additive map $h : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ such that $f(x) = \alpha x + h(x)$ for all $x \in \mathcal{R}$ (Theorem 2.1). As applications, a characterization of k-commuting additive maps on prime rings and von Neumann algebras is obtained respectively (Corollaries 2.2–2.3).