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# Binary factorizations of the matrix of all ones

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ABSTRACT

In this paper, we consider the problem of factorizing the $n \times n$ matrix $J_n$ of all ones into the $n \times n$ binary matrices. We show that under some conditions on the factors, these are isomorphic to a row permutation of a De Bruijn matrix. Moreover, we consider in particular the binary roots of $J_n$, i.e. the binary solutions to $A^m = J_n$. On the one hand, we prove that any binary root with minimum rank is isomorphic to a row permutation of a De Bruijn matrix whose row permutation is represented by a block diagonal matrix. On the other hand, we partially solve Hoffman's open problem of characterizing the binary solutions to $A^2 = J_n$ by providing a characterization of the binary solutions to $A^2 = J_n$ with minimum rank. Finally, we provide a class of roots which are isomorphic to a De Bruijn matrix.



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## 1. Introduction

In this paper, we consider the problem of factorizing the $n \times n$ matrix

$$J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

into the binary matrices. Namely, we restrict ourselves to square $n \times n$ factors $A_i$ that have all their elements in $\{0, 1\}$, i.e. that are adjacency matrices of graphs with $n$ nodes. We are thus looking for the solutions of

$$\prod_{i=1}^{m} A_i = A_1 A_2 \dots A_m = J_n$$

and in particular, the case when all the factors are identical, i.e. we investigate the binary solutions to the equation

$$A^m = J_n.$$

The $g$-circulant binary solutions to $A^m = J_n$ were studied through a convenient representation by Hall polynomials [1,3,2,4]. Remind that a matrix is called $g$-circulant if each row is obtained from the previous one by shifting all its elements of $g$ positions to the right. In particular, it has been proved [1] that some $g$-circulant solutions are isomorphic to a De Bruijn matrix, originally defined in [5]. Nowadays, there are very few results [6] about the general binary solutions to $A^m = J_n$. However, these general solutions are of interest in many problems. Indeed, a solution of $A^m = J_n$ is the adjacency matrix of a directed graph for which given any two nodes $u$ and $v$, there is a unique directed path of length $m$ from $u$ to $v$. In [7], it has been shown that these graphs allow to construct a class of algebras. Moreover, in the framework of the finite-time average consensus problem, the binary solutions to $A^m = J_n$ represent all the communication topologies whose interaction strengths are all equal to $1/\sqrt[m]{n}$ and that reach the consensus at time $m$. In particular, the De Bruijn matrices are of this type and have been shown [8] to be one of the quickest strategies to reach the average consensus. In the present paper, we show how the binary roots of $J_n$ with minimum rank are related to the De Bruijn matrices.

The outline of the paper is as follows: in Section 2 we state some properties on the binary roots of $J_n$ and on the De Bruijn matrices, which are well known roots of $J_n$. In Section 3, we study the commuting factors of the matrix with all ones. We prove that under some conditions on the commuting factors, these are isomorphic to a row permutation of a De Bruijn matrix. In Section 4, we prove that any binary root with minimum rank is isomorphic to a row permutation of a De Bruijn matrix, whose row permutation