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Extending a characterization of majorization to infinite dimensions

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ABSTRACT

We consider recent work linking majorization and trumping, two partial orders that have proven useful with respect to the entanglement transformation problem in quantum information, with general Dirichlet polynomials, Mellin transforms, and completely monotone sequences. We extend a basic majorization result to the more physically realistic infinite-dimensional setting through the use of generalized Dirichlet series and Riemann–Stieltjes integrals.

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1. Introduction

The problem of entanglement transformation concerns the ability to transform from one pure state of some composite system to another, using only local operations and classical communication (LOCC). Such manipulations of entangled states have been characterized by way of the partial order of majorization in the finite-dimensional setting [7] and the infinite-dimensional setting [8].

Quantum mechanics is inherently infinite-dimensional by nature; although much work in quantum information theory is done under the restriction of finite dimensions, it is desirable to generalize to the infinite-dimensional setting (such generalizations are often highly non-trivial). In light of an extension of Nielsen's result [7] to the infinite-dimensional setting by way of ϵ-convertibility for LOCC [8], we extend the majorization result of [9] to the infinite-dimensional setting. Herein, we view the characterization of majorization put forward in [9] as a pure math inequality result; consequently, we do not consider the physical ramifications of infinite-dimensional majorization.

There are several definitions for infinite-dimensional majorization; we shall use that discussed in [8] as it best fits the physical descriptions of infinite-dimensional quantum states. That is, since the majorization condition of Nielsen involves vectors of Schmidt coefficients of pure states, the vectors are necessarily in ℓ_1, and therefore there is no need to consider, for example, A. Neumann's definition [6], which allows for vectors in ℓ_∞. Because we are working with positive trace-class operators, the sum of the eigenvalues that we are considering converges to 1, which leads to the promising realization that our Dirichlet series are well-behaved.

2. Dirichlet series, Mellin transforms, and completely monotone functions

Definition 1. Let I be a real interval. A function f is said to be *completely monotone* on I if $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x \in I$ and all $n = 0, 1, 2, \ldots$.

Bernstein's theorem on completely monotone functions states that a necessary and sufficient condition for a function f to be completely monotone on $(0, \infty)$ is that f is the Laplace transform of a positive measure μ:

$$f(s) = \int_0^\infty e^{-st}\, d\mu(t).$$

We recall that the Mellin transform of a function f on $(0, \infty)$ is the function $\phi(s) = \int_0^\infty f(t)t^{s-1}\, dt$. The Mellin and Laplace transforms are closely related: if $f \in L^1((0, \infty))$ is zero outside of $[0, 1]$, then the Mellin transform of $f(x)$ is the Laplace transform of $f(e^{-x})$. It follows that if $f \in L^1((0, \infty))$ is zero outside of $[0, 1]$, then the Mellin transform of f is completely monotone on $(0, \infty)$ if and only if f is non-negative almost everywhere. We will use this fact in our proof of Theorem 2.