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Maps preserving peripheral spectrum of generalized products of operators [☆]

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Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be standard operator algebras on complex Banach spaces X_1 and X_2, respectively. For $k \geqslant 2$, let $(i_1, \ldots, i_m)$ be a sequence with terms chosen from $\{1, \ldots, k\}$, and assume that at least one of the terms in $(i_1, \ldots, i_m)$ appears exactly once. Define the generalized product $T_1 * T_2 * \cdots * T_k = T_{i_1} T_{i_2} \cdots T_{i_m}$ on elements in $\mathcal{A}_i$. Let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map with the range containing all operators of rank at most two. We show that Φ satisfies that $\sigma_\pi(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma_\pi(A_1 * \cdots * A_k)$ for all $A_1, \ldots, A_k$, where $\sigma_\pi(A)$ stands for the peripheral spectrum of A, if and only if Φ is an isomorphism or an anti-isomorphism multiplied by an mth root of unity, and the latter case occurs only if the generalized product is quasi-semi Jordan. If $X_1 = H$ and $X_2 = K$ are complex Hilbert spaces, we characterize also maps preserving the peripheral spectrum of the skew generalized products, and prove that such maps are of the form $A \mapsto cUAU^*$ or $A \mapsto cUA^tU^*$, where $U \in \mathcal{B}(H, K)$ is a unitary operator, $c \in \{1, -1\}$.

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1. Introduction

Linear maps between Banach algebras which preserve the spectrum are extensively studied in connection with a longstanding open problem due to Kaplansky on invertibility preserving linear maps ([1–5,8,11,12] and the references therein). Recently, the study of spectrum preservers without linearity or additivity assumption also attracted attention of researchers. One of interesting topics of this kind concerns the spectrum of products. In [15], Molnár characterized surjective maps Φ on bounded linear operators acting on a Hilbert space preserving the spectrum of the product of operators, i.e., AB and $\Phi(A)\Phi(B)$ always have the same spectrum. A similar question was studied by Huang and Hou in [10] by replacing the spectrum by several spectral functions such as the left spectrum, spectral boundary, etc. Hou, Li and Wong [9] studied further the maps Φ between certain operator algebras preserving the spectrum of a generalized product $T_1 * T_2 * \cdots * T_k$ of low rank operators. Namely, for all operators $T_1, T_2, \ldots, T_k$ of low rank the spectra of $T_1 * T_2 * \cdots * T_k$ and of $\Phi(T_1) * \Phi(T_2) * \cdots * \Phi(T_k)$ are equal. The generalized product is defined as following.

Definition 1.1. Fix a positive integer k and a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal to i_q for all other q. A generalized product for operators $T_1, \ldots, T_k$ is defined by

$$T_1 * T_2 * \cdots * T_k = T_{i_1} T_{i_2} \cdots T_{i_m}, \tag{1.1}$$

m is called the width of the generalized product.

Furthermore, if $(i_1, i_2, \ldots, i_m)$ is symmetric with respect to i_p, we say that $T_1 * T_2 * \cdots * T_k$ is a generalized semi Jordan product; if

$$\begin{aligned}&(i_{p+1}, i_{p+2}, \ldots, i_{m-1}, i_m, i_1, i_2, \ldots, i_{p-1})\\&\quad = (i_{p-1}, i_{p-2}, \ldots, i_2, i_1, i_m, i_{m-1}, \ldots, i_{p+2}, i_{p+1}),\end{aligned} \tag{1.2}$$

we say that $T_1 * T_2 * \cdots * T_k$ is a generalized quasi-semi Jordan product.

Evidently, this definition of generalized product covers the usual product T_1T_2, Jordan semi-triple BAB and the triple one: $\{T_1, T_2, T_3\} = T_1T_2T_3$, etc.; the definition of generalized semi Jordan product covers the Jordan semi-triple BAB and the product like $T_1 * T_2 * T_3 = T_2T_3^2T_1T_3^2T_2$; the definition of generalized quasi-semi Jordan product covers the products like $A * B = B^rAB^s$ and $T_1 * T_2 * T_3 = T_2T_3^2T_1T_3^2T_2^2T_3^3T_2$.

We also remark that the notation $T_1 * T_2 * \cdots * T_k$ is not unique for $T_1, T_2, \ldots, T_k$ because it depends on the choice of the sequence $(i_1, i_2, \ldots, i_m)$. In this paper, we presume that the sequence $(i_1, i_2, \ldots, i_m)$ is fixed throughout the paper for both algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ involved.

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on a complex Banach space X. Recall that a standard operator algebra $\mathcal{A}$ on a complex Banach space X usually