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Preservers of maximally entangled states

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A R T I C L E   I N F O

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A B S T R A C T

The linear structure of the real space spanned by maximally entangled states is investigated, and used to completely characterize those linear maps preserving the set of maximally entangled states on $M_m \otimes M_m$, where M_m denotes the space of $m \times m$ complex matrices. Aside from a degenerate rank one map, such preservers are generated by a change of orthonormal basis in each tensor factor, interchanging the two tensor factors, and the transpose operator.

1. Introduction and notation

Let M_m be the space of $m \times m$ complex matrices, and let H_m be the real space of $m \times m$ (complex) Hermitian matrices. On a finite-dimensional Hilbert space $\mathcal{H}$ of dimension m, a quantum state ρ is simply a density matrix in H_m (that is, ρ is a positive semi-definite $m \times m$ matrix of trace one). A state ρ is said to be *pure* if it has rank one (in other words, ρ is a rank one (orthogonal) projection).

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In quantum information theory, one of the most important concepts is that of entanglement, which occurs when dealing with a multipartite system. We shall restrict our attention to one of the most common cases, a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces of the same dimension m. In this case, a state ρ is said to be *separable* if one can write $\rho = \sum_{i=1}^{r} p_i \rho_i \otimes \sigma_i$ for some states $\rho_i, \sigma_i \in H_m$ and positive scalars p_i summing to one. Otherwise we say the state is *entangled*.

Entanglement is considered a valuable resource, responsible for the power of quantum computing (see [9] for a standard reference) and for applications such as superdense coding (see [3]) and quantum teleportation (see [2]). There are various ways to measure how much entanglement a state has (though for the bipartite case we consider, many of these measures are equivalent); those states possessing maximal entanglement are of particular importance, and a natural question is: what types of state transformations will preserve maximal entanglement? More generally, what linear transformations will map maximally entangled states back to themselves? The answer to this question forms the main result of this paper:

Theorem 1.1. *Let MES denote the set of maximally entangled states in $M_m \otimes M_m$. A linear map $\Phi : Span(MES) \to Span(MES)$ preserves MES if and only if it has one of the following forms:*

1. $A \otimes B \mapsto (U \otimes V)(A \otimes B)^\sigma (U \otimes V)^*$ *for some unitary matrices $U, V \in M_m$.*
2. $A \otimes B \mapsto (U \otimes V)(B \otimes A)^\sigma (U \otimes V)^*$ *for some unitary matrices $U, V \in M_m$.*
3. $X \mapsto (\mathrm{Tr}\, X)\rho$ *for some $\rho \in MES$.*

Here the map $A \mapsto A^\sigma$ denotes either the identity or the transpose map.

Questions of this type have a long history, and fall under the broader purview of linear preserver problems (two useful surveys are [6,7]). Recently there has been work done on finding and classifying linear preservers of various properties or sets related to quantum information theory. One paper of particular relevance is [4], in which the authors classify linear preservers of separable states (a related paper is [8]). However, they work under the more restrictive assumption that the linear map is surjective, an assumption we shall not require. Another paper which considers more specialized maps preserving maximal entanglement is [5].

This paper shall be organized as follows. Section 1 will conclude by introducing some notation. Section 2 will define what a maximally entangled state is, and investigate the real linear span of such states. Section 3 will prepare for the final section by stating and proving a number of technical lemmas. Finally, Section 4 will contain the proof of our main theorem, where we completely characterize the linear maps preserving maximally entangled states on $M_m \otimes M_m$.

We close this section by fixing some additional notation.