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Product distance matrix of a tree with matrix weights

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A B S T R A C T

Let T be a tree on n vertices and let the $n-1$ edges $e_1, e_2, \dots, e_{n-1}$ have weights that are $s \times s$ matrices $W_1, W_2, \dots, W_{n-1}$, respectively. For two vertices i, j, let the unique ordered path between i and j be $p_{i,j} = e_{r_1} e_{r_2} \dots e_{r_k}$. Define the distance between i and j as the $s \times s$ matrix $E_{i,j} = \prod_{p=1}^{k} W_{e_p}$. Consider the $ns \times ns$ matrix D whose (i,j)-th block is the matrix $E_{i,j}$. We give a formula for $\det(D)$ and for its inverse, when it exists. These generalize known results for the product distance matrix when the weights are real numbers.

1. Introduction

Let T be a tree with vertex set $[n] = \{1, 2, \dots, n\}$. Let $D = (d_{i,j})_{1 \leqslant i,j \leqslant n}$ be its distance matrix, i.e. $d_{i,j}$ is the distance between vertices i and j. A classical result of Graham and Pollak [7] is the following.

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Theorem 1. *Let T be a tree on n vertices and let D be its distance matrix. Then* $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$.

Thus, $\det(D)$ only depends on n and is independent of the structure of the tree T. Later, Graham and Lovász [4,6] gave a formula for the inverse of D. Motivated by this, Bapat and Sivasubramanian, building on the work of Bapat, Lat and Pati [3], considered the exponential distance matrix of a tree T. Let the tree T have n vertices and let $e_1, e_2, \dots, e_{n-1}$ be an ordering of its edges. Let edge e_i have weight q_i, $i = 1, \dots, n-1$, where $q_1, \dots, q_{n-1}$ are commuting indeterminates, and define $\mathsf{E}_T = (e_{i,j})$, the exponential distance matrix of T as follows. For vertices i, j, let $p_{i,j}$ be the unique path between i and j. Define $e_{i,j} = \prod_{k \in p_{i,j}} q_k$. Note that $e_{i,j} = e_{j,i}$ as the q_k's commute with each other. By convention, for all i, we set $e_{i,i} = 1$. With this, Bapat and Sivasubramanian [4] showed the following.

Theorem 2. *Let T be a tree on n vertices with edges having weights $q_1, q_2, \dots, q_{n-1}$ and let E be the exponential distance matrix E. Then,* $\det(\mathsf{E}) = \prod_{i=1}^{n-1}(1 - q_i^2)$.

In [4], a slightly more general setup was considered and the inverse of E was also determined.

In this work, we consider the product distance matrix of a tree with matrix weights. The motivation for considering matrix weights may be described as follows. When we consider product distance, it is natural to let the weights be noncommutative, since the edges on a path come with a natural order. The entries of the product distance matrix are then elements of an underlying ring. The formula for the inverse given in Theorem 4 holds in the case of noncommutative weights, even though we have chosen to formulate the result with the weights being matrices which provide a natural example of noncommutative weights. In the case of the formula for the determinant of the product distance matrix, given in Theorem 3, matrix weights are justified since there are difficulties in defining the determinant of a matrix with general noncommutative elements. It is apparent from our results that noncommutative weights do not present any obstacle in obtaining formulas for the determinant and the inverse of distance matrices.

An application of weighted graphs arises naturally in circuit theory, where the graph represents an electrical network, and the weights on the edges are the resistances. Thus the weights are nonnegative numbers. Ando and Bunce [1], motivated by the work of Duffin [5], consider the case where nonnegative weights are replaced by positive semidefinite matrices, and show that certain operator inequalities extend naturally to the more general setting.

In the context of classical distance, matrix weights have been considered by Bapat in [2] where an analogue of Theorem 1 is proved. It is natural to consider a similar setup in the case of product distance.

Thus, we have a tree T on n vertices and each edge e_i has an $s \times s$ matrix weight W_i, $i = 1, 2, \dots, n-1$. The matrices $W_1, \dots, W_{n-1}$ may be over an arbitrary field, or more