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A Jacobi type Christoffel–Darboux formula for multiple orthogonal polynomials of mixed type

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ABSTRACT

An alternative expression for the Christoffel–Darboux formula for multiple orthogonal polynomials of mixed type is derived from the LU factorization of the moment matrix of a given measure and two sets of weights. We use the action of the generalized Jacobi matrix J, also responsible for the recurrence relations, on the linear forms and their duals to obtain the result.

1. Introduction

In this paper we address a natural question that arises from the LU factorization approach to multiple orthogonality [7]. The Gauss decomposition of a Hankel matrix, which plays the role of a moment matrix, leads in the classical case to a natural description

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of algebraic facts regarding orthogonal polynomials on the real line (OPRL) such as recursion relations and Christoffel–Darboux (CD) formula. In that case we have a chain of orthogonal polynomials $\{P_l(x)\}_{l=0}^{\infty}$ of increasing degree l. In [7] we extended that approach to the multiple orthogonality scenario, and the Gauss decomposition of an appropriate moment matrix led to sequences of families of multiple orthogonal polynomials in the real line (MOPRL), $\{Q^{(\mathrm{II},a_1(l))}_{[\vec{\nu}_1(l);\vec{\nu}_2(l-1)]}\}_{l=0}^{\infty}$ and $\{\bar{Q}^{(\mathrm{I},a_1(l))}_{[\vec{\nu}_2(l);\vec{\nu}_1(l-1)]}\}_{l=0}^{\infty}$. These families happen to be biorthogonal, and therefore we will refer to them as biorthogonal sequences of linear forms. The recursion formulae are relations constructed in terms of the linear forms in these sequences. However, the Daems–Kuijlaars Christoffel–Darboux formula given in Proposition 4 – that was re-deduced in [7] by linear algebraic means (Gauss decomposition) and the use of the ABC theorem – was not expressed in terms of linear forms belonging to the mentioned sequences. This situation is rather different to the OPRL case, in that standard scenario of the CD formula, call it the ABC type CD formula, is expressed in terms of orthogonal polynomials in the sequence. The aim of this paper is to show that, within that scheme, we can deduce an alternative but equivalent MOPRL Christoffel–Darboux formula constructed in terms of linear forms in the sequences $\{Q^{(\mathrm{II},a_1(l))}_{[\vec{\nu}_1(l);\vec{\nu}_2(l-1)]}\}_{l=0}^{\infty}$ and $\{\bar{Q}^{(\mathrm{I},a_1(l))}_{[\vec{\nu}_2(l);\vec{\nu}_1(l-1)]}\}_{l=0}^{\infty}$ as in OPRL situation. Besides we are able to find an OPRL type CD formula, expressed in terms solely of elements in the biorthogonal sequences of MOP of mixed type, there are two prices to pay: first, we need, in general, more terms than in the ABC type CD formula for these MOPs and second, we will need to know the coefficients in the recursion relation; i.e., the Jacobi coefficients. We will refer to these type of CD formulae as Jacobi type CD formulae as they are based on the structure of the Jacobi type matrix associated with the biorthogonal sequences which gives their recursion relations.

We must stress that in the OPRL scenario there are many ways to prove the CD formula [26]. In particular, on the one hand we could prove it using the ABC theorem combined with the moment matrix symmetry and on the other hand using the eigen-value properties of the Jacobi matrix. These two approaches – ABC and Jacobi – lead, in this simple situation, to the same result. However, as already mentioned, in the MOP scenario the two approaches lead to different results: the ABC type CD formula (or Daems–Kuijlaars CD formula) and the Jacobi type CD formula.

1.1. Historical background

Simultaneous rational approximation starts back in 1873 when Hermite proved the transcendence of the Euler number in [21]. Later, K. Mahler delivered at the University of Groningen several lectures [24] where he settled down the foundations of this theory, see also [13] and [22]. Simultaneous rational approximation when expressed in terms of Cauchy transforms leads to multiple orthogonality of polynomials. Given an interval $\Delta \subset \mathbb{R}$ of the real line, let $\mathcal{M}(\Delta)$ denote all the finite positive Borel measures with support containing infinitely many points in Δ. Fix $\mu \in \mathcal{M}(\Delta)$, and let us consider a system of weights $\vec{w} = (w_1, \dots, w_p)$ on Δ, with $p \in \mathbb{N}$; i.e. $w_1, \dots, w_p$ being real integrable