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Ordered vector spaces

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A B S T R A C T

This work was motivated by Problem 49-4 from “IMAGE” Issue Number 49, p. 52, Fall 2012 which was proposed by Rajesh Pereira. Seeking the solution to the problem led to the discovery of some interesting results on ordered vector spaces. It is shown that up to an isomorphism, there is only one order on a finite dimensional real vector space of dimension greater than or equal to 2, namely a lexicographic order on the coefficient vectors with respect to certain special bases. These special bases are called non-Archimedean bases. By examining linear preservers on an ordered vector space a solution to problem 49-4 is found as a by-product.

1. A total Perron–Frobenius theorem for total orders

This work was motivated by interest in Problem 49-4 from “IMAGE” Issue Number 49, p. 52, Fall 2012 which was proposed by Rajesh Pereira and solved by Eugene A. Herman. The title for this section of the paper is the same as the problem title.

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Problem 1 *(Problem 49-4).* Let A be an $n \times n$ real matrix. Show that the following are equivalent:
(a) All the eigenvalues of A are real and nonnegative.
(b) There exists a total order $\geq$ on R^n (partial order where any two vectors are comparable) which is preserved by A (i.e. $\underline{x} \in R^n$ and $\underline{x} \geq 0$ implies that $A\underline{x} \geq 0$) and which makes $(R^n, \geq)$ an ordered vector space.

In looking at this problem it is observed that there is a very interesting theory of ordered vector spaces and their linear preservers. In this article we develop this theory and use the results to solve the problem stated above.

In the study of matrices and operators on normed spaces one finds an abundance of information on *partial orders* (preorders) on the relevant vector lattices over the real field. These *partial orders* provide a useful framework for eliciting information about the vector lattices and operators [3]. For example in the theory of non-negative matrices, *partial orders* engendered by convex cones play a prominent role [1]. In contrast there do not appear (at least to the authors) to be many instances in which *total orders* on vector spaces manifest themselves. In this article we explain why this is so.

2. General theory of ordered vector spaces

Definition 2. Let V be a nonempty set and $\gg$ a binary relation on V such that for $\underline{x}, \underline{y}$ and $\underline{z}$ in V the following hold:

(i) $\underline{x} \gg \underline{x}$ (reflexive),
(ii) $\underline{x} \gg \underline{y}$ and $\underline{y} \gg \underline{x} \Rightarrow \underline{x} = \underline{y}$ (anti-symmetric),
(iii) $\underline{x} \gg \underline{y}$ and $\underline{y} \gg \underline{z} \Rightarrow \underline{x} \gg \underline{z}$ (transitive), and
(iv) for any $\underline{x}$, $\underline{y}$ either $\underline{x} \gg \underline{y}$ or $\underline{y} \gg \underline{x}$ (dichotomous). Then $(V, \gg)$ is said to be a linear or totally ordered system and $\gg$ is said to be a *total order* on V.

Definition 3. On R^n, a lexicographic order $\gg_{lex}$ is defined in the following manner. Let $\underline{x} = [x_1, x_2, \cdots, x_n]^t$ and $\underline{y} = [y_1, y_2, \cdots, y_n]^t$ be in R^n. Then $\underline{x} \gg \underline{y}$ if
(a) $\underline{x} = \underline{y}$ or
(b) if $\underline{x} \neq \underline{y}$ and $i_0 = \min\{i : x_i \neq y_i\}$ then $x_{i_0} > y_{i_0}$.

This makes $(R^n, \gg_{lex})$ a totally ordered vector space.

Definition 4. Assume that $(V, \gg)$ is a totally ordered system. Then $(V, \gg)$ is called an *ordered* vector space if the following hold:
(a) if $\underline{x} \gg \underline{y}$ then $\underline{x} + \underline{z} \gg \underline{y} + \underline{z}$ for all $\underline{x}, \underline{y}$ and $\underline{z}$ in V, and
(b) if $\underline{x} \gg \underline{y}$ and c is a positive scalar, then $c\underline{x} \gg c\underline{y}$ for all $\underline{x}, \underline{y} \in V$ [3], page 49.

From this point on it will be assumed that $(V, \gg)$ is an ordered finite dimensional vector space over the real field of dimension $n \geq 2$.