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The Markov Chain Tree Theorem in commutative semirings and the State Reduction Algorithm in commutative semifields

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ABSTRACT

We extend the Markov Chain Tree Theorem to general commutative semirings, and we generalize the State Reduction Algorithm to general commutative semifields. This leads to a new universal algorithm, whose prototype is the State Reduction Algorithm which computes the Markov chain tree vector of a stochastic matrix.

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Commutative semiring
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1. Introduction

The Markov Chain Tree Theorem states that each (row) stochastic matrix A has a left eigenvector x, such that each entry x_i is the sum of the weights of all spanning trees rooted at i and with edges directed towards i. This vector has all components positive if A is irreducible, and it can be 0 in the general case. It can be computed by means of the State Reduction Algorithm formulated independently by Sheskin [26] and Grassmann, Taksar and Heyman [11]; see also Sonin [27] for more information on this.

In the present paper, our main goal is to generalize this algorithm to matrices over commutative semifields, inspired by the ideas of Litvinov et al. [15,17,19]. To this end, let us mention first the tropical mathematics [1,5,12], which is a relatively new branch of mathematics developed over idempotent semirings, of which the tropical semifield, also known as the max algebra, is the most useful example. In one of its equivalent realizations (see Bapat [3]), the max algebra is just the set of nonnegative real numbers equipped with the two operations $a \oplus b = \max(a, b)$ and $a \cdot b = ab$; these operations extend to matrices and vectors in the usual way. Much of the initial development of max algebra was motivated by applications in scheduling and discrete event systems [1,12]. While this original motivation remains, the area is also a fertile source of problems for specialists in combinatorics and other areas of pure mathematics. See, in particular, [16,20].

According to Litvinov and Maslov [15], tropical mathematics (also called idempotent mathematics due to the idempotency law $a \oplus a = a$) can be developed in parallel with traditional mathematics, so that many useful constructions and results can be translated from traditional mathematics to a tropical/idempotent "shadow" and back. Applying this principle to algorithms gives rise to the programme of making some algorithms universal, so that they work in traditional mathematics, tropical mathematics, and over a wider class of semirings.

There is a well-known universal algorithm, which derives from Gaussian elimination without pivoting. This universal version of Gaussian elimination was developed by Backhouse and Carré [2], see also Gondran [10] and Rote [25]. Based on it, Litvinov et al. [15,17,19] formulated a wider concept of a universal algorithm, and discovered some new universal versions of Gaussian elimination for Toeplitz matrices and other special kinds of matrices. The semifield version of the State Reduction Algorithm found in the present paper can be seen as a new development in the framework of those ideas.

The present paper is also a sequel of our earlier work [4], where the Markov Chain Tree Theorem was proved over the max algebra. To this end, we remark that the max-algebraic analogue of probability is known and has been studied, e.g., by Puhalskii [24] as idempotent probability. Our work is also related to the papers of Minoux [22,23]. However, the Markov Chain Tree Theorem established in the present paper is different from the theorem of [22] which establishes a relation between the spanning tree vector