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Extremal properties of tropical eigenvalues and solutions to tropical optimization problems

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A B S T R A C T

An unconstrained optimization problem is formulated in terms of tropical mathematics to minimize a functional that is defined on a vector set by a matrix and calculated through multiplicative conjugate transposition. For some particular cases, the minimum in the problem is known to be equal to the tropical spectral radius of the matrix. We examine the problem in the common setting of a general idempotent semifield. A complete direct solution in a compact vector form is obtained to this problem under fairly general conditions. The result is extended to solve new tropical optimization problems with more general objective functions and inequality constraints. Applications to real-world problems that arise in project scheduling are presented. To illustrate the results obtained, numerical examples are also provided.

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1. Introduction

Tropical (idempotent) mathematics, which focuses on the theory and applications of idempotent semirings, dates back to a few works published in the early 1960s, such as seminal papers by Pandit [1], Cuninghame-Green [2], Giffler [3], Vorob'ev [4] and Romanovskiĭ [5]. Since these works, there have been many new important results obtained in this field and reported in various monographs, including the most recent books by Golan [6], Heidergott et al. [7], Gondran and Minoux [8], Butkovič [9], and in a great number of contributed papers.

Optimization problems that are formulated and solved in the tropical mathematics setting constitute an important research domain within the field. An optimization problem, which is drawn from machine scheduling, arose in the early paper by Cuninghame-Green [2] to minimize a functional defined on a vector set by a given matrix. In terms of the semifield $\mathbb{R}_{\max,+}$ with the usual maximum in the role of addition and arithmetic addition in the role of multiplication, this problem is represented in the form

$$\text{minimize} \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x},$$

where $\boldsymbol{A}$ is a square matrix, $\boldsymbol{x}$ is the unknown vector, $\boldsymbol{x}^{-}$ is the multiplicative conjugate transpose of $\boldsymbol{x}$, and the matrix-vector operations follow the usual rules with the scalar addition and multiplication defined by the semifield.

The solution to the problem is based on a useful property of tropical eigenvalues. It has been shown in [2] that the minimum in the problem coincides with the tropical spectral radius (maximum eigenvalue) of $\boldsymbol{A}$ and is attained at any tropical eigenvector corresponding to this radius. The same results in a different setting were obtained by Engel and Schneider [10].

In recent years, the above optimization problem has appeared in various applied contexts, which motivates further development of related solutions. Specifically, the problem was examined by Elsner and van den Driessche [11,12] in the framework of decision making. Applications to discrete event systems and location analysis were considered in Krivulin [13–16].

In tropical algebra, the spectral radius is given directly by a closed-form expression, which makes the evaluation of the minimum a rather routine task. By contrast, it may be not trivial to represent all solutions to the problem in terms of tropical mathematics in a direct, explicit form that is suitable for further analysis and applications.

Cuninghame-Green [17] proposed a solution that is based on reducing to a linear programming problem. This solution, however, does not offer a direct representation of all solutions in terms of tropical mathematics. Furthermore, Elsner and van den Driessche [11,12] indicated that, in addition to the eigenvectors, there are other vectors, which can also solve the problem. An implicit description of the complete solution set has been suggested in the form of a tropical linear inequality. Though a solution approach has been proposed, this approach provided particular solutions by means of a numerical