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Perturbations of discrete elliptic operators

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Given V a finite set, a self-adjoint operator $\mathcal{K}$ on $\mathcal{C}(V)$ is called elliptic if it is positive semi-definite and its lowest eigenvalue is simple. Therefore, there exists a unique, up to sign, unitary function $\omega \in \mathcal{C}(V)$ satisfying $\mathcal{K}(\omega) = \lambda\omega$ and then, $\mathcal{K}$ is named (λ, ω)-*elliptic*. Clearly, a (λ, ω)-elliptic operator is singular iff $\lambda = 0$. Examples of elliptic operators are the so-called Schrödinger operators on finite connected networks, as well as the signless Laplacian of connected bipartite networks.

A (λ, ω)-elliptic operator, $\mathcal{K}$, defines an automorphism on $\omega^\perp$ whose inverse is called *orthogonal Green operator of* $\mathcal{K}$. We aim here at studying the effect of a perturbation of $\mathcal{K}$ on its orthogonal Green operator. The perturbation here considered is performed by adding a self-adjoint and positive semi-definite operator to $\mathcal{K}$. As particular cases we consider the effect of changing the conductances on semi-definite Schödinger operators on finite connected networks and on the signless Laplacian of connected bipartite networks. The expression obtained for the perturbed network is explicitly given in terms of the orthogonal Green function of the original network.

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1. Introduction

In this paper we study discrete elliptic operators that are a generalization of standard elliptic Schrödinger operators which were the main theme of [2]. By an elliptic Schrödinger operator on a connected network we understand a positive semi-definite operator formed by the combinatorial Laplacian of the network plus a suitable potential.

If the network Γ has order n and its vertices are labelled from 1 to n, the set of Schrödinger operators on Γ is identified with the set of irreducible symmetric Z-matrices of order n whose off-diagonal elements are given by minus the weight, or conductance, of the edges. Moreover, one of these matrices is an M-matrix (a Stieltjes matrix), iff the corresponding Schrödinger operator is positive semi-definite (positive definite). In particular, such a matrix is a weakly diagonally dominant M-matrix; that is, for every row the magnitude of the diagonal entry is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row, iff the potential of its corresponding Schrödinger operator is non-negative. Since a discrete elliptic operator is a positive semi-definite operator such that its lowest eigenvalue is simple, see [1,4], some of the well-known operators, or matrices, associated with a network, other than the combinatorial Laplacian, fall in this framework. For instance, the signless Laplacian of a connected bipartite network is an elliptic operator.

Although we deal here with elliptic operators whose principal part is the signless Laplacian on a bipartite network, we also consider other elliptic operators defined on general networks. In addition, we also study some perturbations of these discrete elliptic operators. Moreover, since any discrete elliptic operator has an associated orthogonal Green operator, see [1] and references therein, we analyze the effect of the perturbation on this operator. Actually, as the orthogonal Green kernel is nothing else but the Moore–Penrose inverse of the matrix associated with the elliptic operator, the goal in this work is to extend the results in [4,5] to a wider class of elliptic operators, including signless Laplacians on bipartite networks which have deserved the attention of many authors see [6,8,9].

The perturbations here considered consist of adding a self-adjoint and positive semi-definite operator or equivalently, a sum of projectors. In [4], the authors showed that for Schrödinger operators this class of perturbations leads to Schrödinger operators on networks on the same set of vertices and whose conductances are non-negative perturbations of the former ones. In this work, we extend this property to a more general class of discrete elliptic operators. Specifically, we pay attention to a generalization of discrete Schrödinger operators whose principal part is singular and positive semi-definite. This class of operators includes the signless Laplacian on bipartite networks.

Let V be a finite set whose cardinality equals n. The space of real valued functions on V is denoted by $\mathcal{C}(V)$ and for any $F \subset V$ the characteristic function of F is denoted by ε_F. So, ε_V is the function whose value is 1 at each vertex and for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the characteristic function of the set $\{x\}$.