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Note on best possible bounds for determinants of matrices close to the identity matrix

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We give upper and lower bounds on the determinant of a small perturbation of the identity matrix. The lower bounds are best possible, and in most cases they are stronger than well-known bounds due to Ostrowski and other authors. The upper bounds are best possible if a skew-Hadamard matrix of the same order exists.

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Table 1
Summary of lower bound results.

Case	Lower bound	Condition
(A) general	$(1-\delta-(n-1)\varepsilon)(1-\delta+\varepsilon)^{n-1}$	$\delta+(n-1)\varepsilon \le 1$
(B) $\delta=\varepsilon$	$1-n\varepsilon$	$n\varepsilon \le 1$
(C) $\delta=0$	$(1-(n-1)\varepsilon)(1+\varepsilon)^{n-1}$	$(n-1)\varepsilon \le 1$

Table 2
Summary of upper bound results.

Case	Upper bound
(A) general	$((1+\delta)^2+(n-1)\varepsilon^2)^{n/2}$
(B) $\delta=\varepsilon$	$(1+2\varepsilon+n\varepsilon^2)^{n/2}$
(C) $\delta=0$	$(1+(n-1)\varepsilon^2)^{n/2}$

1. Introduction

Many bounds on determinants of diagonally dominant matrices have been given in the literature. See, for example, Bhatia and Jain [1], Elsner [5], Horn and Johnson [8], Ipsen and Rehman [9], Ostrowski [11–13], and Price [14]. We consider the case of a matrix $A = I - E$, where I is the $n \times n$ identity matrix and the elements e_{ij} of E are small. Thus, A is "close" to the identity matrix. A more general case, where A is close to a nonsingular diagonal matrix, can be reduced to this case by row and/or column scaling.

To make precise the sense in which E is small, we introduce two non-negative parameters δ and ε, and require

$$|e_{ij}| \le \begin{cases} \delta & \text{if } i = j; \\ \varepsilon & \text{otherwise.} \end{cases}$$

We consider three cases: (A) is the general case, (B) is when $\delta = \varepsilon$, and (C) is when $\delta = 0$. These cases are all of interest. Case (B) is the simplest, and was considered by Ostrowski [13] and others. Case (C) arises naturally if scaling is used to reduce the diagonal elements to 1. Case (A) is an obvious generalization which unifies the cases (B)–(C), and is required to obtain sharp results in some applications where δ and ε have different orders of magnitude.[1]

For the reader's convenience, the lower and upper bound results are summarized in Tables 1–2. A comparison with previously-published bounds is given in Section 2. Our lower bounds are given in Section 3, and the upper bounds in Section 4.

2. Comparison with previous bounds

It is perhaps surprising that we have only found one of the six bounds (cases (A)–(C), lower and upper) in the literature, although their proofs use standard techniques and are not difficult.

[1] For example, in [2, Corollary 5], an optimization problem leads to the choice $\delta \asymp \varepsilon^2$.