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Semi-nonnegative rank for real matrices and its connection to the usual rank

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The present paper introduces the semi-nonnegative rank for real matrices as an alternative to the usual rank. It is shown that the semi-nonnegative rank takes two possible values which are simple functions of the usual rank. Several characterizations of matrices for which the two ranks coincide are given.

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1. Problem statement and motivation

In the sequels, $\mathbb{R}$ denotes the field of real numbers, $M_{n,p}(\mathbb{R})$ denotes the vectorial space of real matrices of order (n, p), and $M_{n,p}(\mathbb{R}_+)$ is the cone of nonnegative matrices. I_k denotes the identity matrix of order k and 1_k, the vector in $\mathbb{R}^k$ where all its components are equal to 1.

Matrix factorization, where a given matrix is represented as a product of two matrices, has attracted a great deal of works both theoretical and applied [9,14]. The rank is one of the most important concepts associated to matrix factorization. Indeed, the rank of a matrix $X \in M_{n,p}(\mathbb{R})$, referred to as usual rank and denoted here by $r_u(X)$, is the smallest integer k for which X can be factorized as:

$$X = AB, \quad A \in M_{n,k}(\mathbb{R}),\ B \in M_{k,p}(\mathbb{R}). \tag{1}$$

Equivalently, the usual rank of a matrix gives the minimal number of rank one matrices, i.e. dyadic products, needed to write the matrix as a sum of dyads [5].

It is important to remark in factorization (1) that the matrices A and B are allowed to have negative entries. Nowadays, it is well known that considering a nonnegativity constraint in both A and B involves the nonnegative rank of a nonnegative matrix and makes the situation more complex [1,8]. The nonnegative rank of a nonnegative matrix $X^+ \in M_{n,p}(\mathbb{R}_+)$, denoted here as $r_+(X^+)$, is the minimum integer k such that X^+ can be factorized as:

$$X^+ = A^+B^+, \quad A^+ \in M_{n,k}(\mathbb{R}_+),\ B^+ \in M_{k,p}(\mathbb{R}_+). \tag{2}$$

It is observed that the factorization (2) appears more restrictive because it involves only nonnegative matrices. The present paper considers an alternative factorization by considering the nonnegativity constraint in (1) only for B. It introduces the semi-nonnegative rank of a matrix $X \in M_{n,p}(\mathbb{R})$ denoted by $r_s(X)$, as the smallest integer k for which X can be factorized as:

$$X = AB^+, \quad A \in M_{n,k}(\mathbb{R}),\ B^+ \in M_{k,p}(\mathbb{R}_+). \tag{3}$$

Two main reasons motivate our interest to study the semi-nonnegative rank. The first resides in the fact that for a given matrix X, its semi-nonnegative rank does not always coincide with its usual rank ($r_s(X) \neq r_u(X)$). Also, for a given nonnegative matrix X^+, its semi-nonnegative rank is different to its nonnegative rank ($r_s(X^+) \neq r_+(X^+)$). A didactical example is given in Fig. 1. These differences allow us to conclude that the three ranks presented above refer to intrinsically different factorizations.

The second main reason which motivates our interest to study the semi-nonnegative rank is related to recent and significant development in the use of matrix factorizations for various clustering tasks [7,12,13,16,17]. Clustering analysis is a common technical