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Contractions of Lie algebras with 2-dimensional generic coadjoint orbits ☆

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We determine all the contractions within the class of finite-dimensional real Lie algebras whose coadjoint orbits have dimensions ≤ 2.

1. Introduction

The notion of contraction of Lie algebras was introduced on physical grounds by Segal [25], Inönü and Wigner [18]: If two physical theories are related by a limiting process then

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the associated invariance groups should also be related by some limiting process called contraction. For instance, classical mechanics is a limiting case of relativistic mechanics and then the Galilei group is a contraction of the Poincaré group.

Contractions of Lie algebras have been investigated by many authors [16,20,24,26] and continue to be a subject of active interest, particularly in connection with the somewhat inverse problem of deforming Lie algebras [4,11,12]. Recently, it was shown that contractions are exactly opposite of the so-called jump deformations, see [13]. This gives a new approach to the difficult problem of finding all the contractions of a given Lie algebra.

Contractions not only link two Lie algebras but also link some objects related to these Lie algebras such as representations, invariants, special functions and quantization mappings [7–10,21], and also coadjoint orbits, which provide the motivation for the present paper, as we will explain directly, below.

The coadjoint orbits of any Lie group G admit G-invariant symplectic structures, and may be regarded as phase spaces acted on by the group G in a Hamiltonian fashion, in the sense of classical mechanics. That well-known observation allows us to regard the Lie groups with 2-dimensional coadjoint orbits as symmetry groups of the simplest nontrivial phase spaces, in some sense. Therefore it is natural to wonder which ones of these symmetry groups can be further contracted. That is precisely the question which we answer in the present paper (see Theorem 2.4), by considering contractions on Lie algebra level (Definitions 2.2 and 2.3).

Note that contractions of any of the aforementioned symmetry groups of the simplest nontrivial phase spaces necessarily belong to the same class of simplest symmetry groups. More precisely, for any finite-dimensional real Lie algebra $\mathfrak{g}$ associated with the simply connected Lie group G, the maximal dimension of the coadjoint orbits of G is an isomorphism invariant that does not increase for any contraction of $\mathfrak{g}$ (see Lemma 3.3 below). Thus, our results can also be regarded as a contribution to understanding the contraction relationships within particular classes of Lie algebras. Here are some samples of problems that were earlier raised on such relationships within various classes of Lie algebras:

- Which are the real Lie algebras that do not admit other contractions than the abelian Lie algebras and themselves? This was answered in [19].
- Which are all the contractions for low-dimensional Lie algebras? This was settled in [23] for the Lie algebras of dimensions ≤ 4 (see also [13]).
- Is it true that within the class of nilpotent Lie algebras, every algebra is a nontrivial contraction of another algebra? This is the Grunewald–O'Halloran conjecture [15] which was recently addressed in [17].

It is noteworthy that the class of Lie algebras investigated by us (the ones with 2-dimensional generic coadjoint orbits) is restricted neither by dimension nor by nilpotency conditions.