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Matrix-tree theorems and discrete path integration

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A B S T R A C T

We calculate characteristic polynomials of operators explicitly presented as polynomials of rank 1 operators. Corollaries of the main result (Theorem 2.3) include a generalization of the Forman's formula for the determinant of the graph Laplacian [6,8], the celebrated Matrix-tree theorem by G. Kirchhoff [9], and some its extensions and analogs, both known (e.g. the Matrix-hypertree theorem by G. Masbaum and A. Vaintrob [10]) and new.

1. Introduction

A celebrated Matrix-tree theorem (MTT) proved by G. Kirchhoff in 1847 [9] has been attracting a constant attention of specialists since then. It was given several new proofs (see e.g. [4] and the bibliography therein), was used in many contexts, sometimes quite

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unexpected ([3], for just an example); there are also many generalizations of the MTT ([1,2,5,6,8,10], to name just a few).

In its classical form, the MTT expresses the principal minor of some $n \times n$-matrix via summation over the set of trees on n numbered vertices. The matrix involved is a weighted sum of the operators $I - s$ where s runs through the set of all reflections in the Coxeter group A_{n-1}. In this article we generalize the MTT allowing any rank 1 operators instead of $I - s$, and any non-commutative polynomial instead of a weighted sum.

The article is organized as follows. In Section 2.1 we formulate and prove the main result, Theorem 2.3. It expresses the characteristic polynomial of an operator M given as a function of rank 1 operators $M_1, \dots, M_N$ by a sort of "discrete path integration". Then we consider two special cases of the theorem: linear polynomials (Section 2.2) and skew-symmetric ones (Section 2.3). In the first case the discrete path integration is reduced to summation over subsets (Corollary 2.4). In the second case we express a Pfaffian of the operator as a sum over the set of pair matchings (a.k.a. dimer structures; Theorem 2.8).

Section 3 contains some applications of the main theorem. In Section 3.1 we use Theorem 2.3 to prove a formula for the characteristic polynomial of the Laplacian of a line bundle on a graph (this formula was first obtained by R. Forman in [6] using a different method). In Section 3.2 we obtain two corollaries of the Forman's formula: the MTT (in [6] and [8] it was derived from Forman's formula as well) and the D-analog of the MTT (Corollary 3.5). In Section 3.3 we consider a discrete Scroedinger operator, which is a generalization of the graph Laplacian, and prove an expression for its characteristic polynomial.

In Section 3.4 we prove two results in a skew-symmetric case: Theorem 3.7, which is a generalization of the Matrix-hypertree theorem of [10], and its D-analog, Theorem 3.9.

2. General results

2.1. The main theorem

Let V be a vector space of dimension n with a scalar product $\langle \cdot, \cdot \rangle$. For $e, \alpha \in V$ denote by $M[e, \alpha]$ the operator

$$M[e, \alpha](v) \stackrel{\text{def}}{=} \langle \alpha, v \rangle e.$$

$M[e, \alpha] : V \to V$ has rank 1 or is zero.

Choose an integer N and consider two sequences of vectors, $e_1, \dots, e_N \in V$ and $\alpha_1, \dots, \alpha_N \in V$. Define then a linear operator $M : V \to V$ as

$$M = P\big(M[e_1, \alpha_1], \dots, M[e_N, \alpha_N]\big) \tag{1}$$

where