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c-Numerical radius isometries on matrix algebras and triangular matrix algebras

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ABSTRACT

Let $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$ and M_n be the set of $n \times n$ complex matrices. For any $A \in M_n$, define the c-numerical range and the c-numerical radius of A by

$$W_c(A) = \left\{ \sum_{i=1}^{n} c_i \langle Ax_i, x_i \rangle : \{x_1, \ldots, x_n\} \text{ is an orthonormal set in } \mathbb{C}^n \right\}$$

and

$$w_c(A) = \max\{|z| : z \in W_c(A)\},$$

respectively. Let $\mathcal{T}_n$ be the set of $n \times n$ upper triangular matrices. When $w_c(\cdot)$ is a norm on M_n, mappings T on M_n (or $\mathcal{T}_n$) satisfying

$$w_c\bigl(T(A) - T(B)\bigr) = w_c(A - B)$$

for all A, B are characterized. As an intermediate step, we also characterize additive c-numerical range preservers on M_n (or $\mathcal{T}_n$).

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1. Introduction

Let M_n be the set of $n \times n$ complex square matrices. The numerical range and the numerical radius of $A \in M_n$ are defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\} \quad \text{and} \quad w(A) = \max\{|z| : z \in W(A)\}$$

respectively. These concepts have applications in pure and applied mathematics. The readers could refer to [10,12–14] for more information. There are many generalizations of the numerical range. When $c = (c_1, \ldots, c_n)^t \in \mathbb{C}^n$, the c-numerical range and the c-numerical radius of $A \in M_n$ are defined by

$$W_c(A) = \left\{\sum_{i=1}^{n} c_i\langle Ax_i, x_i\rangle : \{x_1, \ldots, x_n\} \text{ is an orthonormal set in } \mathbb{C}^n\right\}$$

and

$$w_c(A) = \max\{|z| : z \in W_c(A)\}$$

respectively. All of the above are special cases of C-numerical ranges and C-numerical radii of A. For any $A, C \in M_n$, the C-numerical range and radius of A are defined by

$$W_C(A) = \{\mathrm{tr}(CUAU^*) : U \in M_n \text{ is unitary}\}$$

and

$$w_C(A) = \max\{|z| : z \in W_C(A)\}$$

respectively. The readers may refer to [16] for more information about this subject.

It is always of interest to characterize mappings with some special properties such as leaving certain functions, subsets or relations invariant. If the mappings are assumed to be linear, the problems are often called linear preserver problems. One may see [11, 21,23] and their references for more information. There is a considerable interest about linear preservers of different generalized numerical ranges or radii. A map T is said to be c-numerical range preserving if

$$W_c(T(A)) = W_c(A)$$

for all A in the domain. The readers may refer to [17] for a survey about this topic.

There is also interest in studying preservers with milder conditions than being linear. For example, given a norm $\|\cdot\|$ on a domain X, what is the form of an isometry T (with no linearity assumption but sometimes surjectivity assumption) such that $\|T(A)-T(B)\| = \|A - B\|$ for all $A, B \in X$? Some problems of this type can be found in [1,2,5,8,9].