Linear Algebra and its Applications 466 (2015) 233–240

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Majorization in Euclidean geometry and beyond

Miroslav Fiedler [1]

Academy of Sciences of the Czech Republic, Inst. of Computer Science, Pod vodárenskou věží 2, 182 07 Praha 8, Czech Republic

A R T I C L E I N F O

Article history:
Received 8 October 2014
Accepted 10 October 2014
Available online 27 October 2014
Submitted by R. Brualdi

MSC:
15B51
51N20
26D15

Keywords:
Majorization
Doubly stochastic matrix
Euclidean simplex
Star
Regular simplex
Volume of a simplex

A B S T R A C T

We relate the well known notion of majorization to the behavior of a pair of simplices in a Euclidean n-space. We obtain a geometrical meaning for the determinant of the involved doubly stochastic matrix. Independently, a basic theorem about volumes of simplices contained one in another, even if of different dimensions, is proved. Some related geometric questions are also presented.

1. Introduction

Inspired by the paper [1], the author returned to the topic mentioned already decades ago in [3]. It concerns the close relationship of the well known notion of majorization with simple geometric objects.

E-mail address: fiedler@math.cas.cz.

[1] With institutional support RVO: 67985807.

http://dx.doi.org/10.1016/j.laa.2014.10.015

Let us recall the notion of majorization in the form presented in [1]. A vector $x \in R^n$ is *majorized* by a vector $y \in R^n$, denoted by $x \prec y$, if for each $k = 1, 2, \ldots, n-1$

$$\sum_{i=1}^{k} x_{i:n} \leq \sum_{i=1}^{k} y_{i:n},$$

and

$$\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n};$$

here, the ordered coordinates of the vector x are $x_{1:n} \geq x_{2:n} \geq \ldots \geq x_{n:n}$, and similarly for y.

In addition, we call such a majorization *nontrivial* if the coordinates of the vector y are not all equal.

A basic fact (see [4], Ch. 2) is:

Fact 1. We have $x \prec y$ if and only if there exists a doubly stochastic $n \times n$-matrix D such that $x = Dy$.

Of course, a doubly stochastic matrix is a nonnegative matrix in which all row and column sums are equal to one.

One should observe the basic aspect of majorization that it essentially compares an *unordered n-tuple* of real numbers with another unordered n-tuple of real numbers. One can thus speak about majorization among n-tuples of points on a line or among n-tuples of vectors in a line.

Let us recall now some notions from the classical Euclidean geometry. A Euclidean vector space is the real inner product space R^n of column vectors with the standard inner product $(x, y) = y^T x$. The point Euclidean n-space E_n based on the vector space R^n has as *points* the column $(n+1)$-tuples with last coordinate 1, e.g.,

$$C = [c_1, c_2, \ldots, c_n, 1]^T, \tag{1}$$

and as *vectors* the column $(n+1)$-tuples with last coordinate 0, e.g.,

$$v = [v_1, v_2, \ldots, v_n, 0]^T. \tag{2}$$

The numbers $c_1, c_2, \ldots, c_n$ are *coordinates* of the point C, the numbers $v_1, v_2, \ldots, v_n$ are coordinates of the vector v. Algebraic operations with points are defined as follows: If $A_1, A_2, \ldots, A_m$ are points, $a_1, a_2, \ldots, a_m$ real numbers, then the symbol $a_1 A_1 + a_2 A_2 + \cdots + a_m A_m$ means a point if and only if $\sum_{k=1}^{m} a_k = 1$, and a vector if and only if $\sum_{k=1}^{m} a_k = 0$. In no other case is this symbol defined. The Euclidean distance, or distance, of two points A and B is the length of the vector $A - B$, i.e., for $A = [a_1, a_2, \ldots, a_n, 1]^T$ and $B = [b_1, b_2, \ldots, b_n, 1]^T$, $|A - B| = \sqrt{(A - B, A - B)}$ which is equal to $\sqrt{\sum_{k=1}^{m} (a_k - b_k)^2}$.