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Derivations of a class of Kadison–Singer algebras ☆

Chengjun Hou

College of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

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ABSTRACT

Let $\mathcal{L}$ be a double triangle lattice of projections in a finite von Neumann algebra acting on a separable and complex Hilbert space $\mathcal{K}$. We show that every derivation from the reflexive algebra determined by $\mathcal{L}$ into $B(\mathcal{K})$ is quasi-spatial and automatically continuous. We also obtain that every local derivation on the reflexive algebra is a derivation.

1. Introduction

Let $\mathcal{K}$ be a complex Hilbert space and $B(\mathcal{K})$ the algebra of all bounded linear operators on $\mathcal{K}$. Suppose $\mathcal{A}$ is a subalgebra of $B(\mathcal{K})$ and $\mathcal{M}$ is a subspace of $B(\mathcal{K})$ containing $\mathcal{A}$ and being an $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a derivation, if $\delta(AB) = \delta(A)B + A\delta(B)$ for every A and B in $\mathcal{A}$. The derivation δ is called to be inner (resp., spatial) if there is an operator T in $\mathcal{M}$ (resp., in $B(\mathcal{K})$) such that $\delta(A) = TA - AT$

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E-mail address: cjhou@yzu.edu.cn.

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for every A in $\mathcal{A}$. More generally, if there exists a closed and densely defined operator T on $\mathcal{K}$ and with domain $\mathcal{D}(T)$ such that $A(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$ and $\delta(A)x = (TA - AT)x$ for every A in $\mathcal{A}$ and x in $\mathcal{D}(T)$, then δ is said to be quasi-spatial. Clearly, the quasi-spatiality is a slightly weaker notion. In all cases, the operator T is called an implementation for innerness, spatiality or quasi-spatiality of δ.

One of the main problems in the theory of derivations for operator algebras is to investigate the automatic continuity, innerness, or (quasi-)spatiality of derivations. In 1966, Kadison and Sakai initiated the study of derivations for von Neumann algebras. Then independently proved that every derivation from a von Neumann algebra into itself is inner. The celebrated result on the automatic continuity of derivations on C^*-algebras was given by J. Ringrose, who proved that every derivation from a C^*-algebra into its Banach bimodule is automatically continuous. Since then, many important results on the theory of derivations for operator algebras, especially for (non-selfadjoint) reflexive operator algebras, were obtained by many authors [1,3,6,7,15,17].

Kadison–Singer algebras (or KS-algebras for simplicity), introduced by Ge and Yuan in [4,5], are a new class of non-selfadjoint operator algebras on Hilbert spaces. These algebras combine triangularity, reflexivity and von Neumann algebra properties into one package. More importantly, the invariant projection lattice, called a KS-lattice, of a KS-algebra can be recaptured by a minimal generating property of the lattice in the von Neumann algebra it generates. Hence the introduction of KS-algebras brings connections between selfadjoint and non-selfadjoint theories, so many techniques and tools in von Neumann algebras can be used to study these non-selfadjoint algebras. In [8], we consider the realizations of the double triangle in abstract lattice theory as a lattice $\mathcal{L}$ of orthogonal projections acting on a Hilbert space $\mathcal{K}$, i.e., $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$ with $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$ for different i and j, where P_i's are orthogonal projections on $\mathcal{K}$ and I is the identity operator on $\mathcal{K}$. We proved that, if $\mathcal{L}$ generates a finite von Neumann algebra, then it is not reflexive and the reflexive lattice it generates is topologically homeomorphic to the two-dimensional sphere $\mathbb{S}^2$ (plus two distinct points corresponding to zero and I). In general, the reflexive lattice is a KS-lattice, and the corresponding reflexive algebra is a KS-algebra.

As a continuation of our study on the Hochshcild cohomology group and automorphisms for KS-algebras in [7,2], the purpose of this paper is to investigate the automatic continuity and (quasi-)spatiality of derivations of KS-algebras generated by double triangle lattices of projections in finite von Neumann algebras, defined in [8]. We first use the techniques and theory of unbounded operators affiliated with finite von Neumann algebras to characterize the structure of the class of KS-algebras, and relate them with transitive algebras (on different Hilbert spaces). Then, we show that every derivation on the reflexive algebra is quasi-spatial and automatically continuous. Finally, we prove that every local derivation on the reflexive algebra is a derivation. We remark that, when the double triangle lattice $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$ is strongly closed [11], i.e., one of the vector sums $P_i(\mathcal{K}) + P_j(\mathcal{K})$ $(i \neq j)$ of the ranges of P_1, P_2 and P_3 is closed, Pang and Yang obtained the same results in [17]. In general, the double triangle in this paper is not